

**NASA
Reference
Publication
1174**

September 1986

NASA-RP-1174 19860023291

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Diagrams for Ceramic and Glass
Materials Using Fatigue Data**

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National Aeronautics
and Space Administration

Scientific and Technical
Information Branch

All measurement values are expressed in the International System of Units (SI) in accordance with NASA Policy Directive 2220.4, paragraph 4.

PREFACE

The purpose of this report is to facilitate fracture control in glass and ceramic systems. This report is intended for those materials engineers who are not thoroughly familiar with fatigue testing in glasses and ceramics but who need to develop design-allowable estimates for structures of these materials. Since this type of testing is relatively new and different from the testing in metal systems, a rationale for the test methodology is presented contrasting it with metal systems, which are more familiar to many materials engineers. The procedures developed are applicable to materials with natural processing and fabrication flaws. Although the procedures can be applied to materials with controlled flaws, no attempt is made at addressing advances made in the area of controlled flaw (indentation) testing.

CONTENTS

	<i>Page</i>
PREFACE	iii
INTRODUCTION	1
DISCUSSION	2
ANNOTATED PROGRAMS.....	6
EXAMPLE OF THE USE OF THE PROGRAM.....	17
ACKNOWLEDGMENTS	19
REFERENCES	21
APPENDIX A—SOME EFFECTS OF A DISTRIBUTION IN THE SIZE OF THE LARGEST FLAW IN A MATERIAL.....	23
APPENDIX B—DERIVATION OF THE DYNAMIC FATIGUE EQUATION.....	25
APPENDIX C—DERIVATION OF THE STATIC FATIGUE EQUATION.....	27
APPENDIX D—ADDITIONAL CONCERNS IN DESIGN-ALLOWABLE DEVELOPMENT.....	29
APPENDIX E—DERIVATION OF THE VARIANCE OF $\ln S_1$	31
APPENDIX F—DERIVATION OF COVARIANCE EXPRESSIONS.....	33
APPENDIX G—DERIVATION OF DYNAMIC AND STATIC FATIGUE MEDIAN VALUE ERROR LIMITS NOT ASSUMING A PARTICULAR DISTRIBUTION.....	35
APPENDIX H—DERIVATION OF DYNAMIC AND STATIC FATIGUE HOMOLOGOUS STRESS RATIO ERROR LIMITS NOT ASSUMING A PARTICULAR DISTRIBUTION....	41
APPENDIX I—DYNAMIC AND STATIC FATIGUE ERROR LIMITS ASSUMING WEIBULL DISTRIBUTIONS.....	45

A METHOD FOR DEVELOPING DESIGN DIAGRAMS FOR CERAMIC AND GLASS MATERIALS USING FATIGUE TEST DATA

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Introduction

Determining the safe lifetime of a glass or ceramic structure using fracture mechanics has been the subject of intensive investigation since its successful use in ensuring the reliability of aircraft and spacecraft windows (reference 1). Theory states that the independent strength-related property of these materials is their critical stress-intensity factor and that this material constant is proportional to the product of the applied stress at failure, times the square root of the failure-initiating crack size. Thus, these brittle materials do not possess a single intrinsic strength, but instead, they possess a strength distribution that depends on the distribution of the largest flaws in the material. Since these materials typically experience subcritical crack growth when stressed in tension in the presence of moisture, predicting a safe lifetime for a piece of hardware requires a determination of crack dynamics in the use environment as well as a determination of the largest preexisting crack size in critical areas of the hardware. Therefore, predicting a safe lifetime for these materials requires an analysis of the following three factors: (1) the size of the largest crack in the areas of highest stress in a piece of hardware, (2) how fast the crack grows under stress, and (3) how big the crack can get before it causes failure.

Unfortunately, the first two factors must usually be arrived at indirectly. In contrast to metals, determination of the largest crack in a piece of glass or ceramic is complicated because by the time a crack is observable in these materials, using most presently available nondestructive test techniques, the material is useless as a load-bearing member (Appendix A). Thus, the approach usually followed is to proof-test the hardware, thereby indirectly ensuring that cracks are less than a certain size. This approach assumes that the expression for the material's critical stress-intensity factor is valid and that the cracks do not grow during proof-test unloading. The latter is usually ensured by running the test in a dry environment and unloading rapidly (reference 2).

The next step in lifetime prediction is to determine crack dynamics. Large-crack propagation techniques, so useful for metals, are typically done first. These tests are quick, conservative of material, and not labor intensive. However, their use is complicated by some evidence that the crack propagation constants determined in this manner are not always the same as those derived indirectly by measuring the strength of specimens with small cracks (reference 3). Therefore, crack velocity measuring techniques such as double cantilever beam and double torsion (reference 4) are not always applicable, especially to materials with a polycrystalline microstructure. To circumvent this difficulty, indirect statistical techniques have been developed. These techniques are stress-rate testing (dynamic fatigue) and time-to-failure testing (static fatigue). These tests are usually modulus of rupture measurements, although biaxial tension measurements have some advantages (reference 5).

In a hardware design effort, time is usually a limiting factor. The usual progression is to begin with large-crack propagation techniques and proceed to dynamic fatigue and finally to static fatigue as time permits. The end result of these procedures is to produce a design diagram.

A design diagram is a parametric plot of time to failure versus applied stress. In using these three techniques to produce design diagrams, time-to-failure predictions are extremely sensitive to experimental uncertainty in the fatigue parameters. Statistical analyses have been developed in the literature (references 3, 6, 7, and 8) to assess this uncertainty so that confidence intervals can be applied to time-to-failure estimates in design diagrams. Since this report is written from the point of view of a materials engineer concerned with developing design-allowable estimates, a good deal of emphasis is placed on confidence interval estimates.

This report illustrates how to implement fatigue data to graph design diagrams with error bars using an APL computer program. The error analyses used are called the median analysis and the homologous stress-ratio analysis for dynamic and static fatigue (reference 3). The report uses, as an example, testing that was performed on a machinable glass-ceramic* used structurally in the Energetic Gamma-Ray Experiment Telescope (EGRET) on the Gamma-Ray Observatory (GRO), which is scheduled to be launched in 1988 (references 9 and 10).

Discussion

A design diagram is a parametric plot of lifetime versus applied stress. To arrive at a design diagram, a starting point is the expression for a stress-intensity factor in opening mode I, which is the tensile mode,[†]

$$K_I = Y\sigma a^{1/2} \quad (1)$$

where Y is a geometric constant, taken as $\sqrt{\pi}$ for surface cracks (reference 11), and σ is the far-field stress[‡] applied to a crack of linear dimension, a . Mode I stress intensity is typically the only one considered because glasses and ceramics do not undergo much plastic deformation, and consequently they fail in tension rather than in shear.

In engineering terms, the theory states that these materials break when the product of σ and $a^{1/2}$ exceeds a critical value, K_{IC} (reference 12). Thus, the independent strength-related property of these materials is their critical stress intensity factor given by the equation

$$K_{IC} = Y\sigma_c a_c^{1/2} \quad (2)$$

where σ_c and a_c are the stress and crack size at failure.

According to this theory, a material does not possess a single characteristic strength. It possesses a distribution of strengths that reflects its distribution of cracks or flaws (Appendix A). If the material does not experience subcritical crack growth, measurement of K_{IC} is where materials strength testing ends and therefore a design diagram is not needed. The part is designed, processed, and maintained in service so that the biggest crack in it is small enough to give the required strength. If, however, the material does experience subcritical crack growth under stress, then an estimate must be made as to how long it takes the largest crack in a piece of the material to grow to critical size. In estimating the time required for a crack to grow subcritically, a simple power law relation between subcritical crack velocity, V , and stress intensity, K_I , is most often assumed (reference 2). The relation is

$$V = AK_I^N \quad (3)$$

where A and N are material/environmental constants. When this is done, it has been shown (Appendix B) that fracture stress, S , at a constant stressing rate, $\dot{\sigma}$, is given by the expression

$$S^{N+1} = B(N+1)S_i^{N-2}\dot{\sigma} \quad (4)$$

*Macor™, Corning 9658 glass-ceramic

[†]In contrast to sliding shear, mode II and tearing shear, mode III (reference 11)

[‡]As distinguished from the concentrated stress at the root of the crack tip (references 12 and 13)

where

$$B = 2/[AY^2(N-2)K_{IC}^{N-2}]$$

$$S_i = \text{inert strength}^*$$

Taking the natural logarithm of equation (4) and dividing by $N+1$ linearizes the equation giving the expression

$$\ln S = \frac{1}{N+1} [\ln B + \ln (N+1) + (N-2) \ln S_i + \ln \dot{\sigma}] \quad (5)$$

For a given inert strength, the increase in observed strength, S , as a function of stressing rate, predicted by equation (3) is called dynamic fatigue.[†]

Similarly, it has been shown (Appendix C) that time to failure, t_f , under a constant applied stress, σ_a , is given by the expression

$$t_f = BS_i^{N-2}\sigma_a^{-N} \quad (6)$$

Linearizing equation (6) by taking its natural logarithm gives the expression

$$\ln t_f = \ln B + (N-2) \ln S_i - N \ln \sigma_a \quad (7)$$

For a given inert strength, the delayed failure behavior predicted by equation (6) is called static fatigue, and phenomenologically it is the same as stress rupture in metals.

The fatigue parameters N and B can be directly determined by measuring K_{IC} (which is needed to calculate B) and by using equation (3) to analyze large-crack velocity data (reference 14). Alternatively, since

$$S_i = K_{IC} / Ya_i^{1/2} \quad (8)$$

where a_i is the initial size of the failure-initiating flaw, these constants could be measured directly from fatigue data if there were some way of measuring, on the same specimen, both the initial size of the preexisting flaw that causes failure and the fatigue behavior (i.e., strength for a given stressing rate (equation 5) or time to failure for a given applied stress (equation 7)). Unfortunately, nondestructive test techniques in ceramics have not been developed to the point that the initial, preexisting, flaw size that causes failure can be measured consistently. One way of measuring that flaw size, however, is to break the specimen in such a manner that no subcritical crack growth occurs (i.e., measure its inert strength, S_i). The failure-initiating flaw size is then determined using equation (8).

This technique has led to a determination of the fatigue parameters N and B using equations (5) and (7) by substituting distributions for the variables and assuming that the distributions are equivalent and partitioned in the same way (references 15 and 16). Thus, for example, it is assumed that median values (\hat{S}_i , \hat{t}_f or \hat{S}_i , \hat{S}) of a test sample of nominally identical specimens can be used for S_i , t_f , and S in equations (5) and (7) to arrive at values for N and B . The median value is used since it is particularly suited to tests in which only a fraction of the total number of specimens is broken, such as static fatigue where it is often impractical to wait for all the specimens to break.

Thus, to determine the fatigue parameters N and B using fatigue data, a population of nominally identical test specimens is made. The specimen population is then divided into test samples containing equal numbers of specimens. For a full complement of fatigue tests, this laboratory uses a population of 210 specimens divided into 7 sample sets of 30 specimens each. First, a set of inert strengths are measured, and then the remaining samples are used to measure the median strengths at three stressing rates for dynamic fatigue and three median times to failure at three applied-stress levels for static fatigue.

*Inert strength is the term given to the strength of a ceramic or glass material when there is no subcritical crack growth. This strength is usually measured so quickly that subcritical crack growth does not have a chance to proceed. Otherwise, it is measured in an environment that precludes crack growth (i.e., a very dry or a very cold environment).

†This is in contrast to metals in which the term fatigue is usually reserved for a cyclical process in which a brittle, cold worked zone is created at the crack tip and carried in front of the crack until a critical size is reached at which point fast fracture occurs.

For dynamic fatigue, a least squares plot of $\ln \hat{S}$ versus $\ln \dot{\sigma}$ is made using the median inert strength \hat{S}_i in equation (5). Similarly, for static fatigue, median times to failure are measured at various stress levels, and a least squares plot is made of equation (7) using median values. From these regression analyses, the fatigue parameters N and B can be evaluated from the slopes and intercepts, respectively. With acquisition of the fatigue parameters N and B , equation (7) can be used to create a design diagram. Using this equation, a plot of $\ln t_f$ versus $\ln \sigma_a$ is a series of straight lines with a slope $-N$, which are parametric in S_i or equivalently in initial crack size a_i , using equation (8). Neither of these plots is very useful since the parameters S_i and a_i are not readily determined in a piece of hardware to verify the initial quality of the piece.

However, the inert-strength distribution of glasses and ceramics can usually be approximated by a two-parameter Weibull distribution

$$F = 1 - e^{-\left(\frac{S_i}{S_o}\right)^{m_i}} \quad (9)$$

where S_o and m_i are called the scaling parameter and the Weibull modulus, respectively, and F is the cumulative probability of failure. To evaluate the Weibull parameters m_i and S_o , a least squares plot is made of the logarithmic restatement of equation (9):

$$x_F = m_i \ln S_i - m_i \ln S_o \quad (10)$$

where $x_F = \ln \ln (1/1-F)$. Equation (7) can then be restated as a design diagram, parametric in probability of failure, by solving equation (10) for $\ln S_i$ and substituting it into equation (7), which gives the expression

$$\ln t_f = \ln B + \left(\frac{N-2}{m_i}\right) [x_F + m_i \ln S_o] - N \ln \sigma_a \quad (11)$$

This is the form of the design diagram that has been found most useful for initiating design. Since the equation does not take into account differences in the stress distribution between test specimens and the hardware, or size differences, it is truly an initial step in an evolving and iterative process as discussed in Appendix D.

Although the median value technique for analyzing fatigue data is quite straightforward to apply, it makes inefficient use of the data since the least squares analysis uses only one value per sample of 20 to 30 specimens. Recognizing the need for more efficient utilization of fatigue data, several researchers (references 17 and 18) have suggested a method of data reduction that is based on homologous stress ratios σ_{HS} and σ_{HD} (reference 19) defined by

$$\sigma_{HS} = \frac{\sigma_a}{S_i} \quad \text{and} \quad \sigma_{HD} = \frac{S}{S_i} \quad (12)$$

Rewriting equations (5) and (7) in terms of these ratios gives

$$\ln \sigma_{HD} = \frac{1}{N+1} \left[\ln B + \ln (N+1) + \ln \frac{\dot{\sigma}}{S_i^3} \right] \quad (13)$$

and

$$\ln (t_f S_i^2) = \ln B - N \ln \sigma_{HS} \quad (14)$$

By ranking the S data from weakest to strongest for a given $\dot{\sigma}$ and then by pairing these data with equally ranked S_i data, a least squares analysis of $\ln \sigma_{HD}$ versus $\ln (\dot{\sigma}/S_i^3)$ will give N and B from the slope and intercept, respectively. Similarly, from a least squares analysis of $\ln (t_f S_i^2)$ versus $\ln \sigma_{HS}$, the parameters N and B are determined from the slope and intercept, respectively, of the static fatigue data. Note that times to failure are ranked from shortest to longest and are paired with increasingly ranked inert strengths from a sample of specimens whose number equals the number of specimens under load in the static fatigue test. If it is necessary, starting sample sizes are made equal by using a random number generator to discard the excess number of specimens.

Since all the data are used rather than just one value per sample of 20 to 30 specimens, this method increases the confidence in N and B as compared with the median value technique (reference 7). The main advantage of using a homologous ratio analysis, however, has been that it points out anomalous behavior, particularly for static fatigue data. An example of this appears in the section discussing the use of the program.

Time-to-failure predictions using equation (11) are very sensitive to uncertainty in the size of the failure-initiating crack and uncertainty in how fast that crack grows. This is reflected in experimental uncertainty in S_i and the fatigue constants N and B . Expressions for this experimental uncertainty and its effect on the uncertainty of $\ln t_f$ can be resolved (references 3 and 6). Determining these expressions entails application of the error-propagation rule (references 3, 6, 20, and 21) to equation (7). The error-propagation rule gives the uncertainty of a quantity in terms of the uncertainty of the variables used to calculate that quantity. In engineering terms, the rule states that if x is a quantity that is a function of at least two other variables μ and ν (i.e., $x = f(\mu, \nu, \dots)$), then the uncertainty of x in terms of its variance $V(x)$ is given in terms of the variance and covariance of the variables μ and ν by the expression

$$V(x) = V(\mu) \left(\frac{\partial x}{\partial \mu} \right)^2 + V(\nu) \left(\frac{\partial x}{\partial \nu} \right)^2 + 2 \text{Cov}(\mu, \nu) \left(\frac{\partial x}{\partial \mu} \right) \left(\frac{\partial x}{\partial \nu} \right) + \dots, \quad (15)$$

there being additional terms and cross terms for additional variables. Since the standard deviation, σ , of x is the square root of the variance of x , a plot of x with 95 percent reproducibility limits would be the value of $x \pm 1.96$ times the standard deviation of x . Similarly, the 90 percent reproducibility limits would be ± 1.645 standard deviations, and so on (reference 3).

Application of the error-propagation rule to $\ln t_f$ in equation (7) gives the expression

$$\begin{aligned} V(\ln t_f) = & \left(\frac{\partial \ln t_f}{\partial B} \right)^2 V(\ln B) + \left(\frac{\partial \ln t_f}{\partial N} \right)^2 V(N) + \left(\frac{\partial \ln t_f}{\partial S_i} \right)^2 V(\ln S_i) \\ & + 2 \text{Cov}(N, \ln B) \left(\frac{\partial \ln t_f}{\partial N} \right) \left(\frac{\partial \ln t_f}{\partial \ln B} \right) \end{aligned} \quad (16)$$

Since σ_a is supposed to be known exactly, terms involving the variance and covariance of $\ln \sigma_a$ have been eliminated from equation (16). Furthermore, since N and B are functionally unrelated to S_i , covariance terms between S_i and those variables are also zero. Evaluating the partial derivatives in equation (16) results in the expression

$$\begin{aligned} V(\ln t_f) = & V(\ln B) + \left(\ln \frac{S_i}{\sigma_a} \right)^2 V(N) + (N-2)^2 V(\ln S_i) \\ & + 2 \text{Cov}(N, \ln B) \ln \left(\frac{S_i}{\sigma_a} \right) \end{aligned} \quad (17)$$

In fatigue testing, expressions for $V(\ln B)$, $V(N)$, $\text{Cov}(N, \ln B)$, and $V(\ln S_i)$ can be found in a number of ways, depending on the assumptions made concerning the type of distribution describing t_f , S , and S_i , and on the type of analysis used (i.e., median or homologous stress-ratio analysis). An expression for $V(\ln S_i)$ is derived in Appendix E, assuming that S_i is described by a two-parameter Weibull distribution. Alternatively, $V(\ln S_i)$ can be calculated directly from the measured inert-strength data using the commonly accepted expression for variance (reference 20)

$$V(\ln S_i) = \sigma^2 = \frac{1}{n-1} \sum_{j=1}^n [(\ln S_i)_j - \overline{(\ln S_i)}]^2 \quad (18)$$

where $\overline{(\ln S_i)}$ is the mean value for $\ln S_i$ (i.e., $1/n \sum_{j=1}^n (\ln S_i)_j$). Either expression for $V(\ln S_i)$ can be used in the median strength analysis or the homologous stress-ratio analysis. The other three terms in equation (17) can be calculated in two ways for the median analysis, but only the first of these ways can be used to calculate these terms for the homologous stress-ratio analysis as shown below.

Since both the median and homologous ratio techniques deal with linear equations, the method for arriving at expressions for $V(\ln B)$, $V(N)$, and $\text{Cov}(N, \ln B)$ can be generalized in terms of the usual equation for a straight line:

$$y = ax + b \quad (19)$$

where y and x are the dependent and independent variables, respectively, while a and b are the slope and intercept.

In linear regression or least squares analysis, it is possible to calculate the variance of the least squares slope and intercept parameters as well as their covariance by using the J -paired values of the dependent and independent variables using the expressions (references 3, 20, 21, and Appendix F):

$$V(a) = \frac{\sigma^2}{JR(x)} \quad (20)$$

$$V(b) = V(a) [R(x) + \bar{x}^2] \quad (21)$$

$$Cov(a, b) = -V(a) \bar{x} \quad (22)$$

where

$$\sigma^2 = \frac{1}{J-2} \sum_{i=1}^J [y_i - (ax_i + b)]^2$$

$$R(x) = \frac{1}{J} \sum_{i=1}^J (x_i - \bar{x})^2$$

$$\bar{x} = \frac{1}{J} \sum_{i=1}^J x_i$$

In the first method for arriving at expressions for $V(\ln B)$, $V(N)$, and $Cov(N, \ln B)$, the error-propagation rule is applied to expressions for a and b after they have been solved for N and $\ln B$, respectively. This gives equations for $V(N)$ and $V(\ln B)$ in terms of $V(a)$ and $V(b)$. The covariance of N and $\ln B$ is then evaluated in terms of $Cov(a, b)$ by using the differential expression for covariance found in reference 3 and developed in Appendix F. This process is performed in Appendix G for median analysis and in Appendix H for the homologous stress-ratio analysis.

The second method for arriving at expressions for $V(\ln B)$, $V(N)$, and $Cov(N, \ln B)$ applies only to the median analysis. The method assumes that t_f , S , and S_i are all described by two-parameter Weibull distributions. Making that assumption, it is possible to derive expressions for the parent sample variance, σ^2 , for the median fracture strength \hat{S} , and the median time to failure \hat{t}_f , in terms of the number of specimens tested, the Weibull modulus, and the fatigue parameter, N (reference 3). In Appendix I, these expressions for $V(\ln B)$, $V(N)$, and $Cov(N, \ln B)$ are reproduced from reference 3 for convenience.

The first method of error-bar estimation (i.e., using only measured data) is preferred for median analysis rather than the second method (i.e., assuming Weibull distributions, for reasons outlined in Appendix E).

Annotated Programs

The analysis just reviewed has been incorporated into an APL computer program called *DESIGN*. The program uses two functions available in APL public libraries and three functions developed to use with *DESIGN* as subroutines. These functions are *LTF*, *SVLTF*, and *DVLTF*. The public library functions are *PCF* and *GRAPH*, which are found in reference 22.

The function *PCF* is a least squares routine that is used for determining the slope and intercept of the linear equations used to evaluate Weibull parameters and fatigue constants. In the program *DESIGN*, the output of *PCF* is a two-element vector: The first element is a slope, and the second an intercept.

The function *GRAPH* is used to plot the design diagrams. Two types of diagrams are developed by *DESIGN*: One diagram is a plot of $\ln t_f$ versus $\ln \sigma_a$ for various probabilities of failure without error bars. The other is a plot of $\ln t_f$ versus $\ln \sigma_a$ for a single probability of failure with error bars.

In the program *DESIGN*, mnemonic abbreviations are used wherever possible. The letter "V" after a designated variable such as applied stress "AS," time to failure "TF," log-applied stress "LAS," failure stress "FS," strength "S," etc., indicates a vector. The letter "V" appearing before a variable such as log time to failure "LTF," fatigue parameters "N" and "B," or log inert strength "LSI" indicates the variance of that variable. The letters "S" and

“D” at the beginning of a function or variable indicates static or dynamic fatigue. Thus, *VLTF* means “variance log time to failure,” whereas *SVLTF* means “variance log time to failure using static fatigue data,” and *DVLTF* means the same thing using dynamic fatigue data.

The commands *SADDESIGN*← 16, 93, 203; *SADVLTF*←9; and *SASVLTF*←9 are stored in the work space named *FATIGUE* in order to regulate the program and allow data input. *DESIGN* is intended to be an interactive program in which the user can compare the results obtained by using different test methods and data reduction schemes. When the program stops before line 93, the results of the Weibull analysis are obtained by typing either *WEIBULL* or *MI*, *SO*. Similarly, when the program stops before line 203, the values of *N* and *B* are obtained by typing these letters or the words *STATIC* or *DYNAMIC*, whichever is appropriate. Values for *V(N)*, *V(ln B)*, *V(ln S_p)*, and *Cov(N, ln B)* are printed automatically during the execution of the functions *SVLTF* and *DVLTF*.

A sequence of operations in the execution of *DESIGN*, *SVLTF*, *DVLTF*, and *LTF* is listed in Tables 1 through 4. Following the tables, the programs are annotated in terms of the equations used in the text of this report.

Table 1
Operational Flow of *DESIGN*

Step	Operation Performed
1	Receive instruction about entering inert-strength data (lines 1-15).
2	Stop before line 16 to enter inert-strength data.
3	Calculate Weibull parameters m_i and S_o (lines 16-23).
4	Calculate variance log inert strength, not assuming a particular distribution, using measured inert-strength data (lines 24-25).
5	Receive instruction about entering fatigue data (lines 26-86).
6	Calculate median inert strength assuming a Weibull distribution (line 87).
7	Initialize <i>x</i> and <i>y</i> values for a least squares determination of slope and intercept (lines 88-92).
8	Stop before line 93 to enter fatigue data.
9	Calculate <i>N</i> and <i>B</i> using static fatigue median value data (lines 93-109).
10	Calculate <i>N</i> and <i>B</i> using static fatigue homologous stress-ratio data (lines 110-130).
11	Calculate <i>N</i> and <i>B</i> using dynamic fatigue median value data (lines 131-147).
12	Calculate <i>N</i> and <i>B</i> using dynamic fatigue homologous stress-ratio data (lines 148-166).
13	Receive instruction on formatting <i>GRAPH</i> to produce design diagrams (lines 167-196).
14	Receive instruction on producing a table of graphed data (lines 197-201).
15	Stop before line 203 to format <i>GRAPH</i> .
16	Graph design diagrams.

Table 2
Operational Flow of *SVLTF*

Step	Operation Performed
1	Receive instruction (lines 1-8).
2	Stop before line 9 to decide the type of error analysis to be performed.
3	Calculate $V(N)$, $V(\ln B)$, $V(\ln S_p)$, and $Cov(N, \ln B)$ assuming Weibull distributions and median value analysis (lines 9-27).
4	Calculate $V(N)$, $V(\ln B)$, and $Cov(N, \ln B)$ using median value analysis but not assuming a particular distribution (lines 29-31).
5	Calculate $V(N)$, $V(\ln B)$, and $Cov(N, \ln B)$ using homologous ratio analysis (lines 33-35).
6	Print $V(N)$, $V(\ln B)$, $Cov(N, \ln B)$, and $V(\ln S_p)$.
7	Calculate variance log time to failure (line 46).

Table 3
Operational Flow of *DVLTF*

Step	Operation Performed
1	Receive instruction (lines 1-8).
2	Stop before line 9 to decide the type of error analysis to be performed.
3	Calculate $V(N)$, $V(\ln B)$, $V(\ln S_p)$, and $Cov(N, \ln B)$ assuming Weibull distributions and median value analysis (lines 9-29).
4	Calculate $V(N)$, $V(\ln B)$, and $Cov(N, \ln B)$ using median value analysis but not assuming a particular distribution (lines 31-34).
5	Calculate $V(N)$, $V(\ln B)$, and $Cov(N, \ln B)$ using homologous ratio analysis (lines 36-39).
6	Print $V(N)$, $V(\ln B)$, $Cov(N, \ln B)$, and $V(\ln S_p)$.
7	Calculate variance log time to failure (line 50).

Table 4
Operational Flow of *LTF*

Step	Operation Performed
1	Calculate inert strength (line 1).
2	Calculate log time to failure (line 2).

▽DESIGN[□]▽
 ▽DESIGN;FV;X1V;Y1V;J3;X1;WEIBULL;SIM;Y2V;X2V;X2;STATIC;Y3V;X3V;X3;DYNAMIC;C90;LL;UL;X
 [1] 'SI UNITS ARE USED FOR DATA ENTRY. STRENGTHS ARE IN'
 [2] 'PASCAL UNITS AND TIME IS IN SECONDS'
 [3] 'ENTER INERT STRENGTH DATA STARTING'
 [4] 'WITH THE LOWEST STRENGTH S(1) AND ENDING WITH THE '
 [5] 'HIGHEST STRENGTH S(K) WHERE K IS THE NUMBER OF'
 [6] 'SPECIMENS, FOR EXAMPLE'
 [7] 'S1 ← 10.183E7'
 [8] ','
 [9] ','
 [10] ','
 [11] 'S25 ← 11.596E7'
 [12] 'END THE DATA INPUT BY ASSIGNING A VALUE TO THE '
 [13] 'VECTOR SV AS FOR EXAMPLE'
 [14] 'SV ← S1,S2,S3,...S25'
 Value assigned to inert-strength vector SV = S1, S2,...S25
 [15] 'FINALLY TYPE THE COMMAND → C1 TO CONTINUE THE PROGRAM'
 [16] C1:FV ← ((1/ρSV) - 0.5) + ρSV
 Value assigned to probability vector FV = (1-0.5)/n, (2-0.5)/n, ..., (n-0.5)/n
 [17] X1V ← • SV
 X1V = ln SV
 [18] Y1V ← • • 1 + 1 - FV
 Y1V = ln ln (1/1 - FV)
 [19] J3 ← 2, ρSV
 J3 = a two-element vector: first element 2, second element ρSV (25 in this example)
 [20] X1 ← J3 ρ X1V, Y1V
 X1 = a two-row, 25-column matrix (in this example): first row X1V, second row Y1V
 [21] WEIBULL ← 1 PCF X1
 WEIBULL = a two-element vector containing slope and intercept of text equation (10)
 [22] MI ← WEIBULL[1]
 MI = m_i = first element of the WEIBULL vector
 [23] S0 ← • - (WEIBULL[2]) + MI
 S0 = S₀ = e^{-(1/m_i) × (intercept of text equation 10)}
 [24] SUM ← +/X1V

$$SUM = \sum_{j=1}^n (\ln S_j)_j$$

 [25] VLSI ← (+/(X1V - SUM + (ρSV)*2) + ((ρSV) - 1))

$$VLSI = V(\ln S_i) = \frac{1}{n-1} \sum_{j=1}^n [(\ln S_j)_j - (\overline{\ln S_i})]^2$$

 [26] 'ENTER FATIGUE DATA: IF MEDIAN VALUE ANALYSIS IS BEING USED,'
 [27] 'THEN THE DATA IS ENTERED IN ORDERED PAIRS BY EITHER ASSIGNING '
 [28] 'VALUES TO THE MEDIAN TIME TO FAILURE VECTOR, TFVM, AND '
 [29] 'TO THE APPLIED STRESS VECTOR, ASV, FOR STATIC FATIGUE'
 [30] 'OR ELSE ASSIGNING VALUES TO THE MEDIAN FAILURE STRESS '
 [31] 'VECTOR, FSV, AND TO THE APPLIED STRESS RATE VECTOR, ARV,'
 [32] 'FOR DYNAMIC FATIGUE, FOR EXAMPLE'
 [33] 'TFVM ← TFM1,TFM2,...'
 [34] 'ASV ← AS1,AS2,...'
 [35] 'WHERE TFM1 IS THE MEDIAN TIME TO FAILURE AT APPLIED '
 [36] 'STRESS AS1, OR '
 [37] 'FSVM ← FSM1,FSM2,...'
 [38] 'ARV ← AR1,AR2,...'
 [39] 'WHERE FSM1 IS THE MEDIAN FAILURE STRESS AT '
 [40] 'APPLIED STRESS RATE AR1'
 [41] ', '
 [42] 'IF HOMOLOGOUS RATIO ANALYSIS IS BEING USED FOR '
 [43] 'STATIC FATIGUE, THEN ENTER THE TIME TO FAILURE VALUES '

[44] 'AS A MATRIX TFX. THE ROWS OF TFX ARE THE VECTORS TFXH1,TFXH2,...ETC.'

[45] 'ASSOCIATED WITH THE APPLIED STRESSES AS1,AS2,...ETC. AT WHICH THE'

[46] 'FAILURE TIMES WERE MEASURED. THE ELEMENTS OF THE VECTORS TFXH(I)'

[47] 'ARE RANKED IN THE ORDER OF INCREASING TIMES TO FAILURE. THE'

[48] 'DIMENSIONS OF THE VECTORS TFXH(I) ARE THE NUMBER OF SPECIMENS TESTED'

[49] 'AT EACH APPLIED STRESS. IF ALL THE SPECIMENS DID NOT FAIL,'

[50] 'ZEROS,(0), ARE PUT IN FOR THEIR TIME TO FAILURE. IF THE'

[51] 'DIMENSION OF THE VECTORS TFXH(I) ARE STILL NOT EQUAL DUE TO'

[52] 'DIFFERENCES IN THE NUMBER OF SPECIMENS TESTED AT EACH APPLIED'

[53] 'STRESS, THEN MAKE THEM EQUAL BY FILLING THE LAST ELEMENTS OF THE'

[54] 'SMALLER DIMENSIONED VECTORS WITH ONES,(1). A SMALL DIMENSIONED'

[55] 'VECTOR MAY THUS CONSIST OF A SERIES OF INCREASING TIMES TO'

[56] 'FAILURE FOLLOWED BY A SERIES OF ZEROS FOLLOWED BY A SERIES OF ONES, FOR EXAMPLE,'

[57] 'TFXH1 ← TF1,TF2,TF3,...0,0,0,...1,1,1'

[58] 'AS1 ← VALUE OF THE FIRST APPLIED STRESS'

[59] 'ASV ← AS1,AS2,...'

[60] 'TFX ← ((ρ ASV),(ρ TFXH1)) ρ TFXH1,TFXH2,...'

[61] 'WHERE TF1 IS THE SHORTEST TIME TO FAILURE AT APPLIED'

[62] 'STRESS AS1 AND TF2 IS THE NEXT SHORTEST TIME ETC. OR,'

[63] ' , '

[64] 'IF HOMOLOGOUS RATIO ANALYSIS IS BEING USED FOR DYNAMIC FATIGUE,'

[65] 'THEN ENTER THE FAILURE STRENGTH VALUES AS A MATRIX FSX.'

[66] 'THE ROWS OF FSX ARE THE VECTORS FSVH1,FSXH2,...ETC.'

[67] 'ASSOCIATED WITH THE APPLIED STRESS RATE AR1,AR2,...ETC.'

[68] 'AT WHICH THE STRENGTHS WERE MEASURED. THE ELEMENTS OF THE'

[69] 'VECTORS FSVH(I) ARE RANKED IN THE ORDER OF INCREASING'

[70] 'STRENGTH. IF THE DIMENSIONS OF THE VECTORS FSVH(I) ARE'

[71] 'NOT EQUAL, THEN MAKE THEM EQUAL BY FILLING THE LAST'

[72] 'ELEMENTS OF THE SMALLER DIMENSIONED VECTORS WITH ZEROS,(0).'

[73] 'FOR EXAMPLE,'

[74] 'FSXH1 ← FS1,FS2,...,FS22,0,0,0'

[75] 'ARI ← VALUE OF THE FIRST APPLIED STRESS RATE'

[76] 'ARV ← AR1,AR2,...'

[77] 'FSX ← ((ρ ARV),(ρ FSVH1)) ρ FSVH1,FSXH2,...'

[78] 'WHERE FS1 IS THE LOWEST FAILURE STRENGTH AT APPLIED'

[79] 'STRESS RATE AR1 AND FS2 IS THE NEXT LOWEST FAILURE'

[80] 'STRENGTH, ETC.'

[81] ' , '

[82] 'AFTER THE DATA INPUT CONTINUE THE PROGRAM BY TYPING'

[83] 'THE COMMAND → STM, → DYM, → STH OR → DYH DEPENDING ON WHETHER STATIC'

[84] 'FATIGUE OR DYNAMIC FATIGUE DATA WITH A MEDIAN ANALYSIS OR STATIC'

[85] 'FATIGUE OR DYNAMIC FATIGUE WITH A HOMOLOGOUS RATIO ANALYSIS ARE'

[86] 'BEING USED TO GENERATE THE FATIGUE PARAMETERS N AND B'

[87] $SIM ← ((MI) × 2) + S0$

[88] $SIM = \hat{S}_i = e^{\ln S_0 + \frac{1}{m_i} \ln \ln 2}$

[89] I ← 1

[90] I = 1, control variable

[91] X2V ← 10 } initialize

[92] Y2V ← 10 } variables

[93] X3V ← 10 } value

[94] Y3V ← 10 }

[95] SIM:Y2V ← TFVM

[96] Y2V = $\ln \hat{t}_f$

[97] X2V ← ASV

[98] X2V = $\ln \sigma_a$

[99] SLS:J3 ← 2, ρX2V

[100] J3 = a two-element vector: first element 2, second element ρX2V

[101] X2 ← J3 ρX2V,Y2V

[102] X2 = a two-row, multicolumn matrix: first row X2V, second row Y2V

slope	intercept
$\ln \hat{t}_f = \underbrace{-N \ln \sigma_a}_{\text{slope}} + \underbrace{\ln B + (N-2) \ln \hat{S}_i}_{\text{intercept}}$	text equation (7) with median values
$\ln (t_f \hat{S}_i^2) = \underbrace{-N \ln \sigma_{HS}}_{\text{slope}} + \underbrace{\ln B}_{\text{intercept}}$	text equation (14)

[97] $STATIC \leftarrow 1 \text{ PCF } X2$
Static = a two-element vector containing slope and intercept of text equation (7) or (14)

[98] $N \leftarrow -STATIC[1]$
 $N = -\text{slope}$

[99] $\rightarrow (I > 1) / BHS$
Branch statement differentiating between calculation of B using homologous ratios ($I > 1$) or median values ($I = 1$).

[100] $B \leftarrow *STATIC[2] - (N - 2) \times \text{SIM}$
 $B = e^{(\text{intercept}) - (N-2) \ln \bar{S}_i}$

[101] $\rightarrow (I = 1) / VS$
Skip the next statement that calculates B for homologous ratio analysis and start calculating data variances.

[102] $BHS: B \leftarrow *STATIC[2]$
 $B = e^{(\text{intercept})} = e^{\ln B}$

[103] $VS: VY2 \leftarrow (+ / ((Y2V - STATIC[2] + X2V \times STATIC[1]) * 2)) + ((\rho Y2V) - 2)$
 $VY2 = V(y) = (1/J - 2) \sum_{i=1}^J [y_i - (ax_i + b)]^2$

[104] $SUM \leftarrow + / X2V$
 $SUM = \sum_{i=1}^J x_i$

[105] $VX2 \leftarrow (+ / X2V - SUM + (\rho Y2V) * 2) + (\rho Y2V)$
 $VX2 = R(x) = (1/J) \sum_{i=1}^J (x_i - \bar{x})^2$

[106] $VST1 \leftarrow VY2 + VX2 \times (\rho Y2V)$
 $VST1 = V(a_2) \text{ or } V(a_4) = 1/J V(y)/R(x)$

[107] $VST2 \leftarrow VST1 \times VX2 + (SUM + (\rho Y2V) * 2)$
 $VST2 = V(b_2) \text{ or } V(b_4) = VST1[R(x) - \bar{x}^2]$

[108] $COV12 \leftarrow -VST1 \times SUM + (\rho Y2V)$
 $COV12 = \text{Cov}(a_2, b_2) \text{ or } \text{Cov}(a_4, b_4) = -(VST1) \bar{x}$

[109] $\rightarrow CT$
Go to instruction for formatting the graph function.

[110] $STH: R \leftarrow \rho ASV$
 $R = \text{control variable equal to the dimension of the applied stress vector}$

[111] $SLP: TFXVH \leftarrow TFX[I;]$
 $TFVH = \text{a vector whose elements are the } I\text{th row of the TFX matrix}$

[112] $TFVH \leftarrow (1 \neq TFXVH) / TFXVH$
 $TFVH$ is compressed to eliminate ones from the vector.

[113] $SVI \leftarrow SV$
Define initial value of inert-stress vector SVI.

[114] $SL: \rightarrow ((\rho TFXVH) = (\rho SVI)) / SE$
Branch statement comparing the dimension of the time-to-failure vector with that of the inert-strength vector.

[115] $\rightarrow ((\rho TFXVH) > (\rho SVI)) / TFG$
Branch statement preparatory to making the dimension of TFXVH and SVI equal.

[116] $DP \leftarrow ?(\rho SVI)$
 $DP = \text{a random number between 1 and } \rho SVI$

[117] $SVI[DP] \leftarrow 0$
 $SVI[DP] = 0$

[118] $SVI \leftarrow (0 \neq SVI) / SVI$
 SVI is compressed to eliminate zeros.

[119] $\rightarrow SL$
Go to the discrimination statement for the vector equilibration loop for static fatigue.

[120] $TFG: DP \leftarrow ?(\rho TFXVH)$
 $DF = \text{a random number between 1 and } \rho TFXVH$

[121] $TFVH[DP] \leftarrow 1$
 $TFVH[DP] = 1$

[122] $TFVH \leftarrow (1 \neq TFXVH) / TFXVH$
 $TFVH$ is compressed to eliminate ones.

[123] $\rightarrow SL$
Go to the discrimination statement for the vector equilibration loop for static fatigue.

- [124] $SE:TFVH \leftarrow (0 \neq TFVH)/TFVH$
TFVH is compressed to eliminate zeros.
- [125] $SVI \leftarrow (\rho TFVH) \setminus SV$
SVI is shortened to the first $\rho TFVH$ elements.
- [126] $Y2V \leftarrow Y2V, \odot TFVH \times SVI * 2$
 $y = \ln(t_f S_i^2)$
- [127] $X2V \leftarrow X2V, \odot ASV[I] + SVI$
 $x = \ln \sigma_d / S_i$
- [128] $I \leftarrow I + 1$
Increase control variable by one.
- [129] $\rightarrow (I \leq R)/SLP$
Branch statement to go through loop again ($I \leq R$) or go to least squares routine ($I > R$).
- [130] $\rightarrow SLS$
Go to least squares routine for static fatigue.
- [131] $DYM:Y3V \leftarrow \odot FSVM$
 $Y3V = y = \ln \hat{S}$
- [132] $X3V \leftarrow \odot ARV$
 $X3V = x = \ln \hat{\sigma}$
- [133] $DLS:J3 \leftarrow 2, \rho X3V$
 $J3 = a \text{ two-element vector: the first element of which is two and the second element } \rho X3V$
- [134] $X3 \leftarrow J3 \rho X3V, Y3V$
 $X3 = a \text{ two-row, multicolumn matrix: the first row } X3V, \text{ second row } Y3V$
- [135] $DYNAMIC \leftarrow 1 PCF X3$
DYNAMIC = a two-element vector containing the slope and intercept of text equation (5) or (13)
- [136] $N \leftarrow (+DYNAMIC[1]) - 1$
 $N = (1/\text{slope} - 1)$
- [137] $\rightarrow (I > 1)/BHD$
Branch statement differentiating between the calculation of B using homologous ratios ($I > 1$) or median values ($I = 1$).
- [138] $B \leftarrow *((N+1) \times DYNAMIC[2]) - (\odot N+1) + (N-2) \times \odot SIM$
 $B = e^{(N+1) \times (\text{intercept}) - [\ln(N+1) + (N-2) \ln \hat{S}_i]}$
- [139] $\rightarrow (I = 1)/VD$
Skip the next statement that calculates B for homologous ratio data and start calculating data variances.
- [140] $BHD:B \leftarrow *((N+1) \times DYNAMIC[2]) - \odot N+1$
 $B = e^{(N+1) \times (\text{intercept}) - \ln(N+1)}$
- [141] $VD:VY3 \leftarrow (+/((Y3V - DYNAMIC[2]) + X3V \times DYNAMIC[1]) * 2) + ((\rho Y3V) - 2)$

$$VY3 = (1/J - 2) \sum_{i=1}^J [y_i - (ax_i + b)]^2$$
- [142] $SUM \leftarrow +/X3V$

$$SUM = \sum_{i=1}^J x_i$$
- [143] $VX3 \leftarrow (+/(X3V - SUM + (\rho Y3V) * 2) + (\rho Y3V))$

$$VX3 = R(x) = (1/J) \sum_{i=1}^J (x_i - \bar{x})^2$$
- [144] $VDY1 \leftarrow VY3 + VX3 \times (\rho Y3V)$
 $VDY1 = V(a_1) \text{ or } V(a_3) = (1/J) V(y)/R(x)$
- [145] $VDY2 \leftarrow VDY1 \times VX3 + (SUM + (\rho Y3V) * 2)$
 $VDY2 = V(b_1) \text{ or } V(b_3) = VDY1[R(x) + \bar{x}^2]$
- [146] $COV12 \leftarrow -VDY1 \times SUM + (\rho Y3V)$
 $COV12 = Cov(a_1, b_1) = Cov(a_3, b_3) = -(VDY1) \bar{x}$
- [147] $\rightarrow CT$
Go to instructions for formatting the graph function.
- [148] $DYH:R \leftarrow \rho ARV$
 $R = \text{control variable whose value equals the dimension of the applied stress vector}$
- [149] $DLP:FSVH \leftarrow FSX[I;]$
 $FSVH = a \text{ vector whose elements are the } I\text{th row of the FSX matrix}$
- [150] $SVI \leftarrow SV$
 $SVI = SV \text{ defines initial value of the inert-strength vector}$

$$\ln \hat{S} = \underbrace{\frac{1}{N+1} \ln \hat{\sigma}}_{X3V \text{ slope } Y3V} + \underbrace{\frac{1}{N+1} [\ln B + \ln(N+1) + (N-2) \ln \hat{S}_i]}_{\text{intercept}}, \text{ text equation (5) with median values}$$

$$\ln \sigma_{HD} = \underbrace{\frac{1}{N+1} \ln \frac{\hat{\sigma}}{S_i^3}}_{\text{slope}} + \underbrace{\frac{1}{N+1} [\ln B + \ln(N+1)]}_{\text{intercept}}, \text{ text equation (13)}$$

[151] $FSVH \leftarrow (0 \neq FSVH)/FSVH$
FSVH is compressed to eliminate zeros.

[152] $DL: \rightarrow ((\rho FSVH) = (\rho SVI))/DE$
Branch statement comparing the dimension of the failure stress vector with that of the inert-strength vector.

[153] $\rightarrow ((\rho FSVH) > (\rho SVI))/FSG$
Branch statement preparatory to making the dimension of FSVH and SVI equal.

[154] $DP \leftarrow ?(\rho SVI)$
DP = a random number between 1 and ρSVI

[155] $SVI[DP] \leftarrow 0$
SVI[DP] = 0

[156] $SVI \leftarrow (0 \neq SVI)/SVI$
SVI is compressed to eliminate zeros.

[157] $\rightarrow DL$
Go to the discrimination statement for the vector equilibration loop for the dynamic fatigue.

[158] $FSG: DP \leftarrow ?(\rho FSVH)$
DP = a random number between 1 and $\rho FSVH$

[159] $FSVH[DP] \leftarrow 1$
FSVH[DP] = 1

[160] $FSVH \leftarrow (1 \neq FSVH)/FSVH$
FSVH is compressed to eliminate ones.

[161] $\rightarrow DL$
Go to the discrimination statement for the vector equilibration loop.

[162] $DE: Y3V \leftarrow Y3V, \oplus FSVH + SVI$
 $Y3V = y = \ln S/S_i$

[163] $X3V \leftarrow X3V, \oplus ARV[I] + SVI * 3$
 $X3V = x = \ln \sigma/S_i^3$

[164] $I \leftarrow I + 1$
Loop control variable update.

[165] $\rightarrow (I \leq R)/DLP$
Control variable comparison branch statement.

[166] $\rightarrow DLS$
Go to least squares routine for dynamic fatigue.

[167] **CT: 'DECIDE ON THE LENGTH OF THE LN(APPLIED STRESS) AXIS BY'**

[168] **'ASSIGNING A VALUE TO THE VECTOR LASV. FOR EXAMPLE'**

[169] **'LASV $\leftarrow 16.45 + 0.05 \times 140$ '**

[170] **'BEGINS THE AXIS AT 16.50 WITH A SCALE OF 0.05 FOR 40'**

[171] **'POINTS, THUS ENDING THE AXIS AT 18.45'**

[172] **' '**

[173] **'NEXT DECIDE WHICH TYPE OF DESIGN DIAGRAM IS DESIRED.'**

[174] **'FOR MULTIPLE PROBABILITIES WITHOUT CONFIDENCE LIMITS,'**

[175] **'SPECIFY FAILURE PROBABILITIES AND PERFORM THE FUNCTION'**

[176] **'LTF AFTER EACH SPECIFICATION, STORING THE RESULT IN'**

[177] **'AN APPROPRIATELY NAMED VARIABLE, FOR EXAMPLE'**

[178] **'F $\leftarrow .001$ '**

[179] **'LTF001 $\leftarrow LTF$ '**

[180] **'FOR A DESIGN DIAGRAM AT A SPECIFIC FAILURE PROBABILITY'**

[181] **'WITH CONFIDENCE LIMITS SPECIFY THE FAILURE PROBABILITY'**

[182] **'AND PERFORM EITHER THE FUNCTIONS LTF AND SVLTF OR LTF'**

[183] **'AND DVLTF AFTER THE SPECIFICATION, STORING THE RESULT IN AN'**

[184] **'APPROPRIATELY NAMED VECTOR, FOR EXAMPLE'**

[185] **'F $\leftarrow .001$ '**

[186] **'LTF001 $\leftarrow LTF$ '**

[187] **'DVLTF001 $\leftarrow DVLTF$ '**

[188] **' '**

[189] **'NEXT ENTER THE ARGUMENT FOR THE GRAPH FUNCTION, FOR EXAMPLE'**

[190] **'X $\leftarrow 7.40 \rho LASV, LTF001, LTF01, LTF1, LTF5, LTF99, LTF999$ '**

[191] **'OR'**

[192] **'C90 $\leftarrow 1.645 \times DVLTF001 * .5$ '**

[193] **'LL $\leftarrow LTF001 - C90$ '**

[194] 'UL ← LTF001 + C90'

[195] 'X ← 4 40 ρ LASV, LL, LTF001, UL'

[196] ' , '

[197] 'IN ORDER TO GENERATE A TABLE OF GRAPHED DATA USE '

[198] 'THE TRANSPOSITION FUNCTION ϕ, FOR EXAMPLE '

[199] 'V ← LASV, LTF001, LTF01, LTF1, LTF5, LTF99, LTF999 '

[200] 'V1 ← 7 40 ρ V '

[201] ' ϕ V1 '

[202] 'FINALLY TO CONTINUE THE PROGRAM TYPE THE COMMAND → C2 '

[203] C2:GRAPH X

▽

▽SVLTF[□]▽

▽VLTF ← SVLTF; KI; A1; A2; A3; KS; LASV1; ALAS; RLAS; VN; COVNLB; VLB; A7; A8; A9; A10; SI; J1; J2; J0

[1] 'IF YOU ARE USING MEDIAN ANALYSIS AND ASSUMING WEIBULL'

[2] 'DISTRIBUTIONS FOR LOG TIME TO FAILURE AND INERT STRENGTH'

[3] 'CONTINUE THE CALCULATION BY TYPING THE COMMAND → W.'

[4] 'IF YOU ARE NOT ASSUMING WEIBULL DISTRIBUTIONS AND ARE USING '

[5] 'ONLY FATIGUE AND STRENGTH DATA TO DEFINE THE ERROR LIMITS ON'

[6] 'FATIGUE) PARAMETERS AND LOG INERT STRENGTH, THEN CONTINUE THE'

[7] 'CALCULATION BY TYPING THE COMMAND → M FOR MEDIAN ANALYSIS OR '

[8] '→ H FOR HOMOLOGOUS RATIO ANALYSIS'

[9] W: J1 ← ρ SV

$J_1 = \text{dimension of the inert-strength vector}$

[10] J2 ← ρ ASV

$J_2 = \text{dimension of the applied stress vector}$

[11] J0 ← J1 × J2

$J_0 = J_1 \times J_2$

[12] KI ← ((MI*2) × ((12 ÷ MI) + ((11 ÷ MI)*2) - 1) + ((ϕ ϕ 2) - MI × ϕ 11 ÷ MI)*2

$$K_i = \frac{m_i^2 \Gamma(1 + 2/m_i) - \Gamma^2(1 + 1/m_i)}{\Gamma^2(1 + 1/m_i)} + [\ln \ln 2 - m_i \ln \Gamma(1 + 1/m_i)]^2$$

[13] A1 ← ((ϕ ϕ 2) - (MI ÷ N - 2) × ϕ 12 + (N - 2) ÷ MI)*2

$$A_1 = \left[\ln \ln 2 - \frac{m_i}{N-2} \ln \Gamma\left(1 + \frac{N-2}{m_i}\right) \right]^2$$

[14] A2 ← 12 × (N - 2) ÷ MI

$$A_2 = \Gamma\left(1 + \frac{2(N-2)}{m_i}\right)$$

[15] A3 ← (1(N - 2) ÷ MI)*2

$$A_3 = \Gamma^2\left(1 + \frac{N-2}{m_i}\right)$$

[16] KS ← (((MI ÷ N - 2)*2) × (A2 + A3) - 1) + A1

$$K_s = (m_i/N-2)^2 \left[\Gamma\left(1 + \frac{2(N-2)}{m_i}\right) - \Gamma^2\left(1 + \frac{N-2}{m_i}\right) \right] / \Gamma^2\left(1 + \frac{N-2}{m_i}\right) + \left[\ln \ln 2 - \frac{m_i}{N-2} \ln \Gamma\left(1 + \frac{N-2}{m_i}\right) \right]^2$$

[17] LASV1 ← ϕ ASV

$LASV1 = \ln \sigma_a$

[18] ALAS ← (+ J2) × + / LASV1

$$ALAS = \overline{\ln \sigma_a} = \frac{1}{J_2} \sum_{j=1}^{J_2} \ln \sigma_{aj}$$

[19] RLAS ← (+ J2) × + / (LASV1 - ALAS)*2

$$RLAS = R(\ln \sigma_{aj}) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \sigma_{aj} - \overline{\ln \sigma_a})^2$$

[20] VN ← (KS × (N - 2)*2) ÷ J0 × (MI*2) × RLAS

$VN = V(N) = K_s (N-2)^2 / J_0 m_i^2 R(\ln \sigma_a)$

[21] COVNLB ← VN × ALAS

$COVNLB = \text{Cov}(N, \ln B) = V(N) \overline{\ln \sigma_a}$

[22] VLB ← (VN × ((ALAS - ϕ SIM)*2) + RLAS) + ((N - 2)*2) × KI ÷ J1 × MI*2

$$VLB = V(\ln B) = V(N) [(\ln \hat{S}_i - \overline{\ln \sigma_a})^2 + R(\ln \sigma_a)] + (N-2) K_i / J_1 m_i^2$$

[23] $A7 \leftarrow ((I2 + MI) \div (I1 \div MI) * 2) - 1$
 $A_7 = [\Gamma(1 + 2/m_i) - \Gamma^2(1 + 1/m_i)] / \Gamma^2(1 + 1/m_i)$

[24] $A8 \leftarrow \text{⊗} \text{⊗} 1 \div 1 - F$
 $A_8 = \ln \ln (1/(1-F)) = x_F$

[25] $A9 \leftarrow \text{⊗} I1 \div MI$
 $A_9 = \ln \Gamma(1 + 1/m_i)$

[26] $A10 \leftarrow (A8 * 2) \div J1 \times MI * 2$
 $A_{10} = x_F^2 (1/J_1 m_i^2)$

[27] $VLSI \leftarrow A10 + ((A7 + A9 * 2) \div J1) - 2 \times A8 \times A9 \div J1 \times MI$

$$VLSI = V(\ln S_i) = x_F^2 (1/J_1 m_i^2) + 1/J_1 \left[\{ [\Gamma(1 + 2/m_i) - \Gamma^2(1 + 1/m_i)] / \Gamma^2(1 + 1/m_i) \} \right. \\ \left. + [\ln \Gamma(1 + 1/m_i)]^2 \right] - (x_F^2 / J_1 m_i) \ln \Gamma(1 + 1/m_i)$$

[28] $\rightarrow C3$

[29] $M:VN \leftarrow VST1$
 $VN = V(N) = VST1 = V(a_2)$

[30] $VLB \leftarrow (((\text{⊗} SIM) * 2) \times VST1) + VST2 + (2 \times (\text{⊗} SIM) \times COV12) + ((2 + STATIC[1]) * 2) \times VLSI \div (\rho SV)$
 $VLB = V(\ln B) = (\ln \hat{S}_i)^2 V(a_2) + V(b_2) + 2(\ln \hat{S}_i) \text{Cov}(a_2, b_2) + ((a_2 + 2) / J_1) V(\ln S_i)$

[31] $COVNLB \leftarrow -COV12$
 $COVNLB = \text{Cov}(N, \ln B) = -\text{Cov}(a_2, b_2)$

[32] $\rightarrow C3$

[33] $H:VN \leftarrow VST1$
 $V(N) = VST1 = V(a_4)$

[34] $VLB \leftarrow VST2$
 $VLB = V(\ln B) = VST2 = V(b_4)$

[35] $COVNLB \leftarrow -COV12$
 $COVNLB = \text{Cov}(N, \ln B) = -\text{Cov}12 = -\text{Cov}(a_4, b_4)$

[36] $C3:SI \leftarrow *((\div MI) \times \text{⊗} \text{⊗} (1 \div 1 - F)) + \text{⊗} S0$
 $SI = S_i = e^{-(1/m_i) \ln \ln (1/(1-F)) + \ln S_0}$

[37] $AS \leftarrow *LASV$

[38] $'VN EQUALS'$

[39] VN

[40] $'VLB EQUALS'$

[41] VLB

[42] $'COVNLB EQUALS'$

[43] $COVNLB$

[44] $'VLSI EQUALS'$

[45] $VLSI$

[46] $VLTF \leftarrow VLB + (VN \times (\text{⊗} SI + AS) * 2) + (VLSI \times (N - 2) * 2) + 2 \times COVNLB \times (\text{⊗} SI + AS)$

$$\nabla VLTF = V(\ln t_f) = V(\ln B) + (\ln S_i / \sigma_d)^2 V(N) + (N - 2)^2 V(\ln S_i) + 2 \text{Cov}(N, \ln B) \ln (S_i / \sigma_d)$$

$\nabla DVLTF[\square] \nabla$

$\nabla VLTF \leftarrow DVLTF; A1; A2; A3; A4; KD; LARV; ALAR; RLAR; AS; VN; A6; COVNLB; VLB; A7; A8; A9; A10; J1; J2; J0; SI$

[1] $'IF YOU ARE USING MEDIAN ANALYSIS AND ASSUMING WEIBULL'$

[2] $'DISTRIBUTIONS FOR LOG TIME TO FAILURE AND INERT STRENGTH,'$

[3] $'CONTINUE THE CALCULATION BY TYPING THE COMMAND $\rightarrow W$.'$

[4] $'IF YOU ARE NOT ASSUMING WEIBULL DISTRIBUTIONS AND ARE USING'$

[5] $'ONLY FATIGUE AND STRENGTH DATA TO DEFINE THE ERROR LIMITS ON'$

[6] $'FATIGUE PARAMETERS AND LOG INERT STRENGTH, THEN CONTINUE THE'$

[7] $'CALCULATION BY TYPING THE COMMAND $\rightarrow M$ FOR MEDIAN ANALYSIS OR $\rightarrow H$ FOR'$

[8] $'HOMOLOGOUS RATIO ANALYSIS.'$

[9] $W:J1 \leftarrow \rho SV$
 $J_1 = \text{dimension of the inert-strength vector}$

[10] $J2 \leftarrow \rho ARV$
 $J_2 = \text{dimension of the applied stress vector}$

[11] $J0 \leftarrow J1 \times J2$
 $J_0 = J_1 \times J_2 = \text{total number of specimens}$

[12] $A1 \leftarrow 1 + (N - 2) \div MI \times N + 1$
 $A_1 = 1 + (N - 2) / m_i (N + 1)$

- [13] $A2 \leftarrow 1 + 2 \times (N-2) \times MI \times N + 1$
 $A_2 = 1 + 2 (N-2)/m_i(N+1)$
- [14] $A3 \leftarrow MI \times (N+1) + N-2$
 $A_3 = m_i (N+1)/(N-2)$
- [15] $A4 \leftarrow ((\otimes \otimes 2) - A3 \times \otimes !A1-1) \times 2$
 $A_4 = \{\ln \ln 2 - m_i [(N+1)/(N-2)] \ln \Gamma [1 + (N-2)/m_i (N+1)]\}^2$
- [16] $KD \leftarrow A4 + (A3 \times 2) \times ((!A2-1) + (!A1-1) \times 2) - 1$
 $K_d = m_i^2 [(N+1)/(N-1)]^2 \{\Gamma (1 + (2/m_i)(N-2)/(N+1)) - \Gamma^2 (1 + (N-2)/m_i(N+1))\} / \Gamma^2 (1 + (N-2)/m_i(N+1))$
- [17] $LARV \leftarrow \otimes ARV$
 $LARV = \ln \hat{\sigma}$
- [18] $ALAR \leftarrow (+J2) \times + / LARV$
 $ALAR = \overline{\ln \hat{\sigma}} = (1/J_2) \sum_{j=1}^{J_2} \ln \hat{\sigma}_j$
- [19] $RLAR \leftarrow (+J2) \times + / (LARV - ALAR) \times 2$
 $RLAR = R (\ln \hat{\sigma}) = (1/J_2) \sum_{j=1}^{J_2} (\ln \hat{\sigma}_j - \overline{\ln \hat{\sigma}})^2$
- [20] $A5 \leftarrow ((N-2) \times (N+1)) \times 2$
 $A_5 = (N-2)^2 (N+1)^2$
- [21] $VN \leftarrow KD \times A5 \div J0 \times RLAR \times MI \times 2$
 $VN = V(N) = K_d (N-2)^2 (N+1)^2 / J_0 m_i^2 R (\ln \hat{\sigma})$
- [22] $A6 \leftarrow (\otimes B \times N + 1) + ALAR - 1 + 3 \times \otimes SIM$
 $A_6 = \ln [B (N+1)] - 3 \ln \hat{S}_i - 1 + \overline{\ln \hat{\sigma}}$
- [23] $COVNLB \leftarrow (VN + N + 1) \times ALAR$
 $COVNLB = \text{Cov} (N, \ln B) = K_d (N-2)^2 (N+1) \overline{\ln \hat{\sigma}} / J_0 m_i^2 R (\ln \hat{\sigma})$
- [24] $VLB \leftarrow (VN \div (N+1) \times 2) \times (RLAR \times 1 + J2) + (-A6) \times 2$
 $VLB = V (\ln B) = \{K_d (N-2)^2 / J_0 m_i^2 R (\ln \hat{\sigma})\} \{[3 \ln \hat{S}_i + 1 - \ln [B (N+1)] - \overline{\ln \hat{\sigma}}]^2 + (1 + J_2) R (\ln \hat{\sigma})\}$
- [25] $A7 \leftarrow ((!2 + MI) \div (!1 + MI) \times 2) - 1$
 $A_7 = [\Gamma (1 + 2/m_i) - \Gamma^2 (1 + 1/m_i)] / \Gamma^2 (1 + 1/m_i)$
- [26] $A8 \leftarrow \otimes \otimes 1 \div 1 - F$
 $A_8 = \ln \ln [1/(1-F)] = x_F$
- [27] $A9 \leftarrow \otimes !1 \div MI$
 $A_9 = \ln \Gamma (1 + 1/m_i)$
- [28] $A10 \leftarrow (A8 \times 2) \div J1 \times MI \times 2$
 $A_{10} = x_F^2 V (1/m_i) = x_F^2 / J_1 m_i^2$
- [29] $VLSI \leftarrow A10 + ((A7 + A9 \times 2) \div J1) - 2 \times A8 \times A9 \div J1 \times MI$
 $VLSI = V (\ln S_i) = x_F^2 V (1/m_i) + V (\ln S_0) + 2 \text{Cov} (m_i, \ln S_0) x_F$
- [30] $\rightarrow C3$
- [31] $M:VN \leftarrow ((+DYNAMIC[1]) \times 4) \times VDY1$
 $VN = V(N) = (1/a_1)^4 V(a_1)$
- [32] $A12 \leftarrow (- (DYNAMIC[2]) \div (DYNAMIC[1]) \times 2) + (\div DYNAMIC[1]) + ((\div DYNAMIC[1]) \times 2) \times \otimes SIM$
 $A_{12} = (-b_1/a_1^2 + 1/a_1 + (\ln \hat{S}_i)/a_1^2)$
- [33] $VLB \leftarrow (((A12) \times 2) \times VDY1) + (((\div DYNAMIC[1]) \times 2) \times VDY2) + (2 \times (A12) \times (\div DYNAMIC[1]) \times COV12) + (((\div DYNAMIC[1]) - 3) \times 2) \times VLSI \div (\rho SV)$
 $VLB = V (\ln B) = (-b_1/a_1^2 + 1/a_1 + (1/a_1^2) \ln \hat{S}_i)^2 V(a_1) + (1/a_1)^2 V(b_1) + 2 (-b_1/a_1^2 + 1/a_1 + (1/a_1^2) \ln \hat{S}_i)(1/a_1) \text{Cov} (a_1, b_1) + 1/J_1 (1/a_1 - 3)^2 V (\ln S_i)$
- [34] $COVNLB \leftarrow -(N+1) \times ((+DYNAMIC[1]) \times 2) \times COV12$
 $COVNLB = \text{Cov} (N, \ln B) = - (N+1)(1/a_1^2) \text{Cov} (a_1, b_1)$
- [35] $\rightarrow C3$
- [36] $H:VN \leftarrow ((+DYNAMIC[1]) \times 4) \times VDY1$
 $VN = V(N) = (1/a_3^4) V(a_3)$
- [37] $A13 \leftarrow (- (DYNAMIC[2]) \div (DYNAMIC[1]) \times 2) + (\div DYNAMIC[1])$
 $A_{13} = (-b_3/a_3^2 + 1/a_3)$
- [38] $VLB \leftarrow (((A13) \times 2) \times VDY1) + (((\div DYNAMIC[1]) \times 2) \times VDY2) + 2 \times (A13) \times (\div DYNAMIC[1]) \times COV12$
 $VLB = V (\ln B) = (-b_3/a_3^2 + 1/a_3)^2 V(a_3) + (1/a_3)^2 V(b_3) + 2 (-b_3/a_3^2 + 1/a_3)(1/a_3) \text{Cov} (a_3, b_3)$
- [39] $COVNLB \leftarrow -((+DYNAMIC[1]) \times 2) \times COV12$
 $COVNLB = \text{Cov} (N, \ln B) = (-1/a_3^2) \text{Cov} (a_3, b_3)$


```

[40]  C3:SI ← * ( ÷ MI ) × * * ( 1 ÷ 1-F ) + * S0
      Si = e-(1/mf) ln ln (1/1-F) + ln S0
[41]  AS ← * LASV
[42]  'VN EQUALS'
[43]  VN
[44]  'VLB EQUALS'
[45]  VLB
[46]  'COVNLB EQUALS'
[47]  COVNLB
[48]  'VLSI EQUALS'
[49]  VLSI
[50]  VLTF ← VLB + ( VN × ( * SI ÷ AS ) * 2 ) + ( VLSI × ( N-2 ) * 2 ) + 2 × COVNLB × ( * SI ÷ AS )
      VLTF = V ( ln tf ) = V ( ln B ) + ( ln Si / σa )2 V ( N ) + ( N-2 )2 V ( ln Si ) + 2 Cov ( N, ln B ) ln ( Si / σa )
      ∇
      ∇LTF[ ] ∇
      ∇R ← LTF; SI
[1]  SI ← * ( ( ÷ MI ) × * * ( 1 ÷ 1-F ) + * S0
      SI = Si = e(1/mf) ln ln (1/1-F) + ln S0
[2]  R ← ( -N × LASV ) + ( * B ) + ( N-2 ) × * SI
      ln tf = - N ln σa + ln B + ( N-2 ) ln Si
      ∇

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Example of the Use of the Program

The program *DESIGN* was used to initiate the iterative process of hardware development for the meter square spark-chamber frames for the EGRET. The frames were made from Corning 9658 machinable glass-ceramic, Macor™. Reports on the static and dynamic fatigue testing of this material appear in references 9 and 10, respectively. This section of the report contains design diagrams for Macor, using a homologous ratio analysis of dynamic fatigue testing data. Two diagrams are presented: One diagram (Figure 1) is a plot of equation (11) for probabilities of failure of 0.001, 0.01, 0.1, 0.5, 0.99, and 0.999. The other (Figure 2) is a plot of equation (11) with error bars at the 90-percent confidence level for a probability of failure of 0.001.

The homologous ratio analysis of static fatigue data for this material was interesting. A plot of the homologous ratio data is shown in Figure 3. The figure indicates that there is an anomaly in a graph of the data around $\ln \sigma_a / S_i$ equal to -0.52. The anomaly is the rapid increase in $\ln (t_f S_i^2)$ at values of $\ln \sigma_a / S_i$ below -0.52. For a material that is susceptible to subcritical crack growth, such a plot is expected to be a straight line with a slope whose negative is the fatigue parameter N (equation 14). The interpretation of this anomaly is that it represents a fatigue limit in this material.

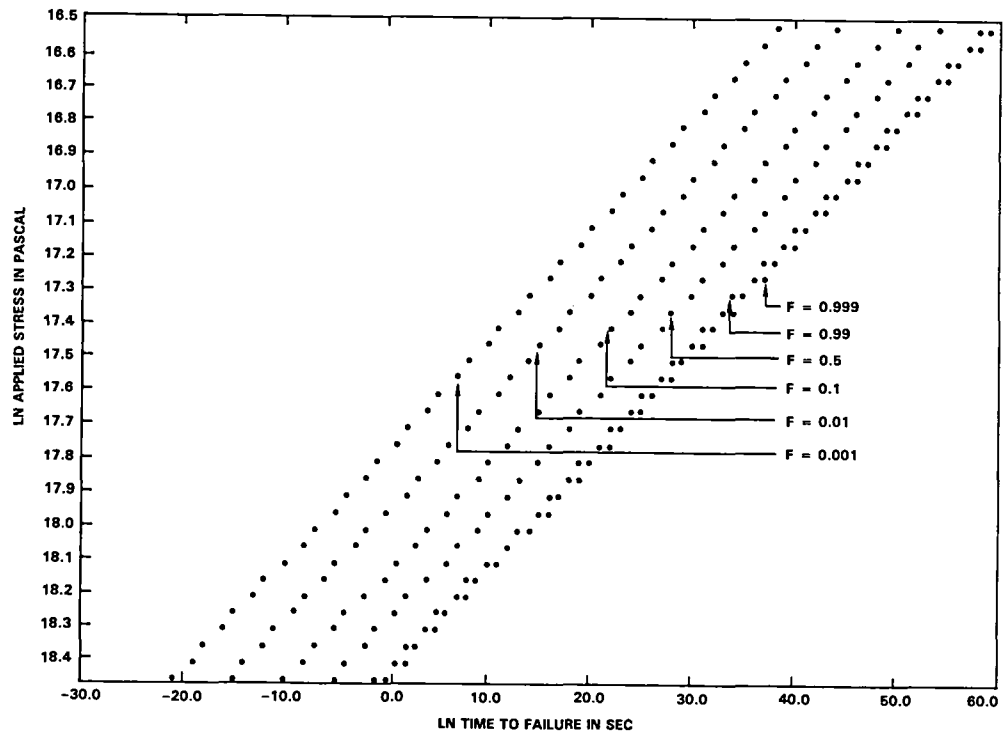


Figure 1. Design diagram for Macor at various probabilities of failure using dynamic fatigue data.

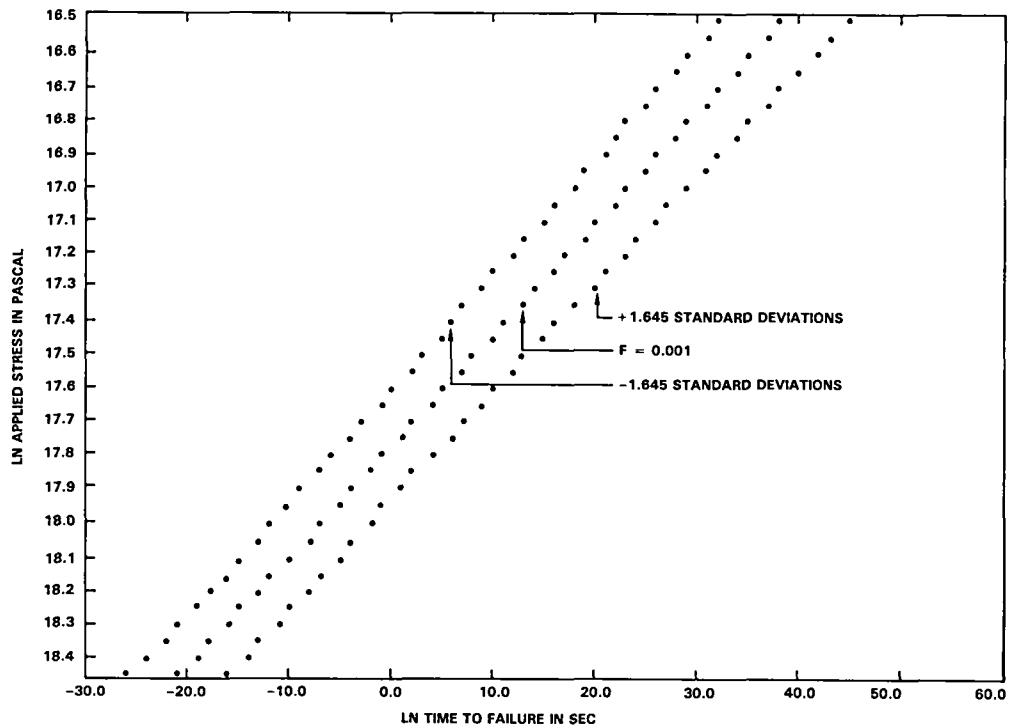


Figure 2. Design diagram for Macor with $F=0.001$ using dynamic fatigue homologous ratio analysis.

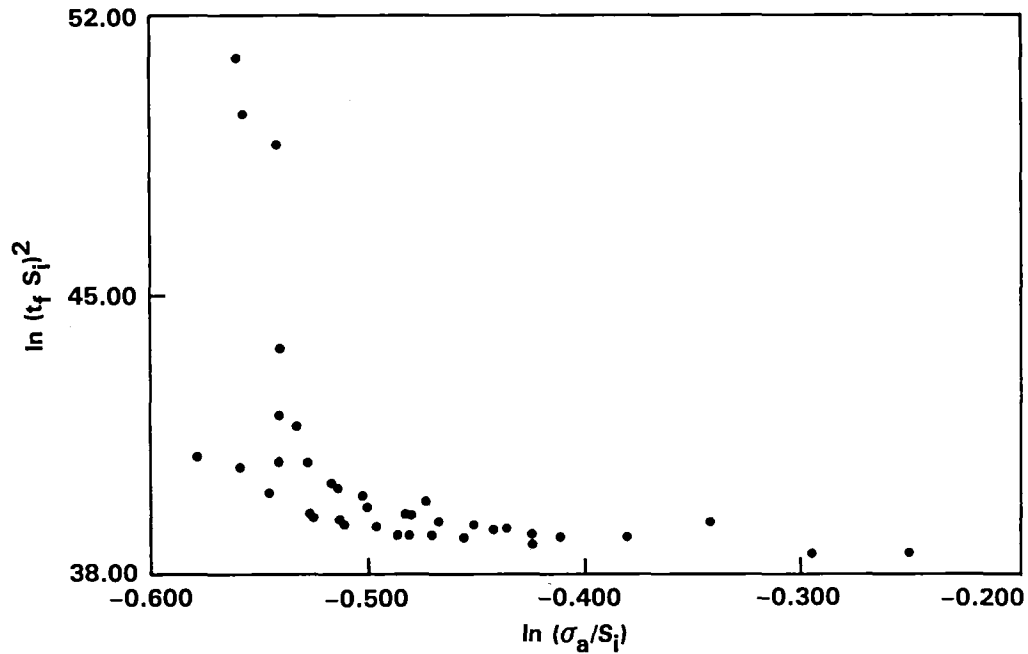


Figure 3. Static fatigue homologous ratio least squares plot for Macor.

Acknowledgments

A large number of people contributed to this work. The authors would like to thank the EGRET project team for making this work possible, particularly Bill Cruickshank, Dave Studenick, Jim Ryan, Earl Angulo, Don Kniffen, and Carl Fichtel. In addition, the authors would like to thank Ron Hunkeler, Becky Derro, Sharon Cooper, Carl Walch, Carl Haehner, and Steve Pagano for their faithful specimen preparation and test implementation efforts. The nondestructive test expertise and advice of Mike Barthelmy and Joanne Uber are gratefully acknowledged. Finally, the authors would like to thank Dr. John Ritter of the University of Massachusetts in Amherst and Drs. Steve Freiman and Sheldon Wiederhorn at the National Bureau of Standard in Gaithersburg, Maryland, for their helpful discussions, advice, and encouragement.

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APPENDIX A

SOME EFFECTS OF A DISTRIBUTION IN THE SIZE OF THE LARGEST FLAW IN A MATERIAL

The strength of high modulus, low K_{IC} materials is very sensitive to surface damage. A plot of failure strength, σ_c , versus the square root of the failure-initiating flaw size, $a_c^{1/2}$, is a hyperbola with the axes as the asymptotes. A plot for K_{IC} equal to values ranging from 0.1 to 10 MPa m^{1/2} is shown in Figure A-1. The plot is intended to illustrate that proof-testing is normally necessary to ensure the quality of glasses and ceramics and that improving surface quality to get higher strengths does not necessarily improve reliability as discussed below.

Normal nondestructive test techniques do not consistently find flaws less than 1 mm ($a_c^{1/2} < 0.03$ m^{1/2}). For metals ($K_{IC} \gg 10$ MPa m^{1/2}), this is normally no problem because corresponding strengths are much greater than 150 MPa and a 10-percent variation in the size of the worst flaw in a piece of material does not appreciably change observed strength. However, for a glass or ceramic material ($K_{IC} \sim 0.5$ to 5 MPa m^{1/2}), flaws > 0.1 mm ($a_c^{1/2} > 0.01$ m^{1/2}) are at the lower size limit of even special NDE, and such flaws result in strengths which are unacceptably low. In addition, a 10-percent variation in the size of the worst flaw in a piece of such material has a major effect on the observed strength of that material since observed strengths lie on the high strength leg of the K_{IC} hyperbola. Therefore, improving processing parameters to increase average strength does not necessarily improve reliability.

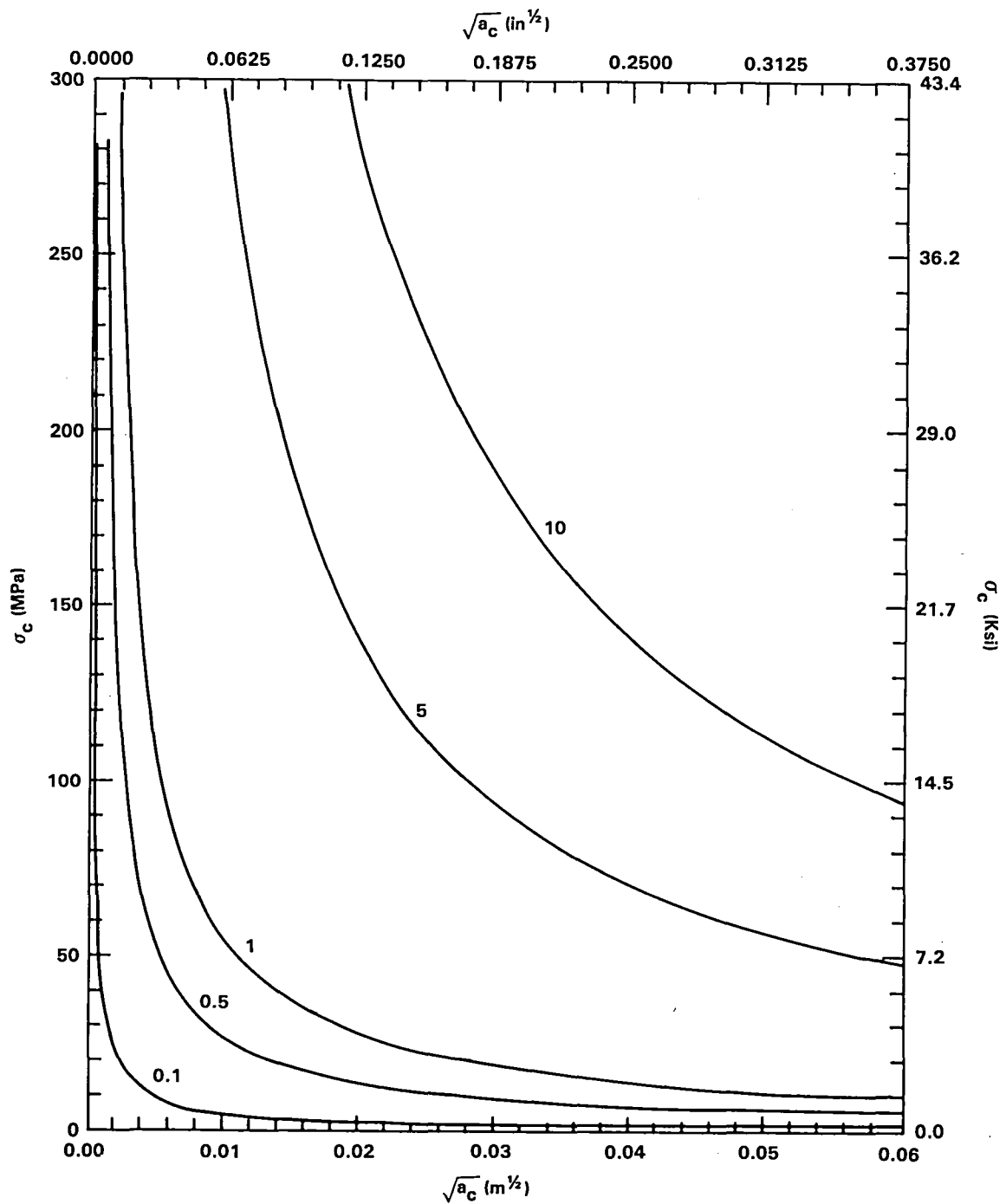


Figure A-1. Failure stress versus square root of crack length for critical stress-intensity factors ($K_{IC} = 0.1, 0.5, 1, 5, \text{ and } 10 \text{ MPa m}^{1/2}$).

APPENDIX B

DERIVATION OF THE DYNAMIC FATIGUE EQUATION

(after reference 19)

$$V = \frac{da}{dt} = AK_1^N, \text{ but } K_1 = Y\sigma\sqrt{a}; \text{ therefore, } V = A(Y\sigma\sqrt{a})^N$$

Rearranging factors, $da/a^{N/2} = AY^N\sigma^N dt$ and recognizing that $\dot{\sigma} = d\sigma/dt$ is a constant and that $dt = d\sigma/\dot{\sigma}$ results in the integral expression

$$\int_{a_i}^{a_f} \frac{da}{a^{N/2}} = \int_{\sigma_i}^{\sigma_f} \frac{AY^N\sigma^N d\sigma}{\dot{\sigma}}$$

which on evaluating gives

$$\left. \frac{2a^{-(\frac{N-2}{2})}}{(2-N)} \right|_{a_i}^{a_f} = \left. \frac{AY^N\sigma^{N+1}}{\dot{\sigma}(N+1)} \right|_{\sigma_i}^{\sigma_f}$$

Neglecting $a_f^{-\frac{N-2}{2}}$ relative to $a_i^{-\frac{N-2}{2}}$ and assuming $\sigma_i = 0$ gives

$$\frac{2}{(N-2)} a_i^{-(\frac{N-2}{2})} = \frac{AY^N\sigma_f^{N+1}}{(N+1)\dot{\sigma}}$$

But a_i is the starting flaw size in the absence of slow crack growth. It is related to inert strength S_i by the expression $a_i = K_{IC}^2/S_i^2 Y^2$. Thus

$$\frac{2}{(N-2)} \left(\frac{K_{IC}^2}{S_i^2 Y^2} \right)^{-(\frac{N-2}{2})} = \frac{AY^N\sigma_f^{N+1}}{(N+1)\dot{\sigma}}$$

but σ_f is observed strength.

Therefore,

$$S^{N+1} = \frac{2(N+1)\dot{\sigma} S_i^{N-2}}{(N-2) AY^2 K_{IC}^{N-2}} = B (N+1) \dot{\sigma} S_i^{N-2}$$

where

$$B = 2/(N-2) AY^2 K_{IC}^{N-2}$$

This equation can also be used to estimate how fast a stressing rate, $\dot{\sigma}$, must be used before subcritical crack growth is eliminated and inert strengths are being measured. The inert-strength stressing rate is

$$\dot{\sigma}_i = S_i^3/B(N+1)$$

APPENDIX C

DERIVATION OF THE STATIC FATIGUE EQUATION (after reference 23)

$V = \frac{da}{dt} = AK_1^N$, but $K_1 = Y\sigma \sqrt{a}$ and $\sigma = \sigma_a$, which is a constant. Thus,

$$V = A(Y\sigma_a \sqrt{a})^N \text{ or } \frac{da}{a^{N/2}} = AY^N \sigma_a^N dt$$

which on integrating gives

$$\int_{a_i}^{a_f} \frac{da}{a^{N/2}} = \int_{t_i}^{t_f} AY^N \sigma_a^N dt$$

Assuming $t_i = 0$ and neglecting $a_f^{N/2}$ relative to $a_i^{N/2}$ gives, on evaluating the integrals,

$$\frac{2a_i^{-(\frac{N-2}{2})}}{(N-2)} = AY^N \sigma_a^N t_f$$

but again as with dynamic fatigue,

$$a_i = K_{IC}^2 / S_i^2 Y^2$$

Therefore,

$$\left(\frac{2}{N-2} \right) \left(\frac{K_{IC}^2}{Y^2 S_i^2} \right)^{-\frac{N-2}{2}} = AY^N \sigma_a^N t_f$$

Solving for t_f gives

$$t_f = \frac{2S_i^{N-2} \sigma_a^{-N}}{AY^2(N-2) K_{IC}^{N-2}} = BS_i^{N-2} \sigma_a^{-N}$$

$$B = \frac{2}{AY^2(N-2) K_{IC}^{N-2}}$$

APPENDIX D

ADDITIONAL CONCERNS IN DESIGN-ALLOWABLE DEVELOPMENT

For materials that do not possess a characteristic strength such as is commonly associated with metals, developing design allowables for a piece of hardware is an evolutionary process between design and materials engineers. It typically starts with an initial design with a certain stress distribution containing a maximum stress that is checked relative to the applied stress given by equation (11) for the required mission lifetime. As it stands, however, equation (11) does not take many things into account that, when included, typically reduce the maximum allowable design stress. It assumes that the hardware will be a relatively small 3- or 4-point bend specimen. To correctly apply equation (11) requires that the anticipated parent flaw population that will cause failure in the hardware is the same as the test specimens (i.e., failure will occur from similar populations of edge, surface, or volume flaws). In addition, differences in the stress distributions between the test specimens and the hardware must be accounted for. Finally, the size difference between the two systems must be accounted for. An example of how this was accomplished for EGRET spark-chamber frames is given in references 9 and 10. Other useful information in this area is found in references 24 through 26.

As previously noted, this type of analysis does not obviate the need for proof-testing. Proof-testing establishes a starting point for lifetime estimates based on crack dynamics derived from fatigue testing. Given proof-test levels and crack propagation data, margins of safety can be calculated.

APPENDIX E

DERIVATION OF THE VARIANCE OF $\ln S_i$ (after reference 3)

$$\ln S_i = \frac{1}{m_i} x_F + \ln S_o$$

$$x_F = \ln \ln \left(\frac{1}{1-F} \right)$$

$$F = \text{the cumulative failure probability} = \frac{i-0.5}{n}$$

i = rank from weakest to strongest

n = number of specimens tested

Applying the error-propagation rule to determine the variance in $\ln S_i$ in terms of the variability of m_i and S_o and assuming that x_F is a constant gives

$$\begin{aligned} V(\ln S_i) &= \left(\frac{\partial \ln S_i}{\partial 1/m_i} \right)^2 V(1/m_i) + \left(\frac{\partial \ln S_i}{\partial \ln S_o} \right)^2 V(\ln S_o) + 2 \text{Cov}(1/m_i, \ln S_o) \left(\frac{\partial \ln S_i}{\partial 1/m_i} \right) \left(\frac{\partial \ln S_i}{\partial \ln S_o} \right) \\ &= x_F^2 V(1/m_i) + V(\ln S_o) + 2 \text{Cov}(1/m_i, \ln S_o) x_F \end{aligned}$$

but from reference 3, page 804, equations (A15), (A26), and (A27), respectively,

$$V(1/m_i) = 1/J_1 m_i^2$$

$$V(\ln S_o) = \frac{1}{J_1} \left\{ \frac{\Gamma(1+2/m_i) - \Gamma^2(1+1/m_i)}{\Gamma^2(1+1/m_i)} + [\ln \Gamma(1+1/m_i)]^2 \right\}$$

$$\text{Cov}(1/m_i, \ln S_o) = -\frac{1}{m_i^2} \text{Cov}(m_i, \ln S_o) = -\frac{1}{m_i^2} \frac{m_i}{J_1} \ln \Gamma(1+1/m_i)$$

In the derivation of the foregoing three expressions, the authors in reference 3 assume that x_F is an independent variable, making equations (20) through (22) applicable to the evaluation of $V(m_i)$ and $V(\ln S_o)$. For what the authors accomplished in reference 3, this is a reasonable assumption because they generated fatigue data for an "ideal" material on a computer using a Monte Carlo technique. It was assumed that the fatigue parameters N and B were known and that the inert-strength distribution of this ideal material was given by a two-parameter Weibull distribution, the modulus and scale parameter of which were known. With the Monte Carlo technique, a given number of specimen strengths were then calculated by randomly selecting their failure probability from a uniform distribution between 0 and 1, making x_F the independent variable. The result is that $V(\ln S_o)$ and $\text{Cov}(m_i, \ln S_o)$ are independent of S and S_o , depending only on the number of specimens J_1 and the Weibull modulus m_i .

It can be argued that in the actual determination of m_i and S_o , x_F is not an independent variable since it is arrived at by simply ranking the measured strength values. The quantity $V(x_F) = (1/J_1 - 1) \Sigma (x_F - \bar{x}_F)^2$ could as easily belong to a sample of strengths with a range from 10 to 20 MPa as from 20 to 30 MPa. If measured strength values are considered, the independent variable and the linear expression of the Weibull equation are rearranged to reflect this. The result is

$$x_F = m_i \ln S_i - m_i \ln S_o$$

Subsequently, going through an analysis similar to that in reference 3 gives expressions that differ from the ones derived previously. Because of this discrepancy and the desire to stay as close to the actual data as possible, error-bar estimation without assuming Weibull distributions is preferred.

A technique sometimes used to do homologous ratio analyses with fatigue samples in which the number of specimens does not match the inert-strength sample size is to calculate Weibull parameters and then generate strengths corresponding to the rankings in the largest sample size. Since this method generates more data points than have actually been measured, it is not preferred. Using only the test data measured to determine the variances of the fatigue parameters without assuming any form of distributions is preferred.

APPENDIX F

DERIVATION OF COVARIANCE EXPRESSIONS

Differential Expression

Reference 3, page 805, indicates

$$\text{Cov} (N, \ln B) = \frac{d \ln B}{dN} V (N)$$

This expression can be derived from the definition of covariance in Bevington (reference 20, page 59) as follows:

$$\sigma_{\mu\nu}^2 \equiv \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma [(\mu_i - \bar{\mu})(\nu_i - \bar{\nu})]$$

Given this definition, it is possible to arrive at a differential expression for covariance as follows:

$$\begin{aligned} \sigma_{\mu\nu}^2 &= \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma \left[\left(\frac{\mu_i - \bar{\mu}}{\nu_i - \bar{\nu}} \right) (\nu_i - \bar{\nu})^2 \right] = \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma \frac{d\mu_i}{d\nu_i} (\nu_i - \bar{\nu})^2 \\ &\approx \left(\lim_{J \rightarrow \infty} \frac{1}{J} \Sigma \frac{d\mu_i}{d\nu_i} \right) \left(\lim_{J \rightarrow \infty} \frac{1}{J} \Sigma (\nu_i - \bar{\nu})^2 \right) = \frac{d\mu}{d\nu} V (\nu) \end{aligned}$$

Linear Regression Expression

Reference 3, page 803, indicates

$$\text{Cov} (a, b) = - \frac{\sigma^2}{JR (x)} \bar{x}$$

This expression is derived similarly to the differential expression for covariance:

$$\sigma_{ab}^2 = \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma \left[\left(\frac{b_i - \bar{b}}{a_i - \bar{a}} \right) (a_i - \bar{a})^2 \right] = \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma \frac{db_i}{da_i} (a_i - \bar{a})^2$$

But from equation (19),

$$b = -ax + y$$

thus,

$$\frac{db_i}{da_i} = -x_i$$

and

$$\sigma_{ab}^2 = - \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma x_i (a_i - \bar{a})^2 \approx \left(- \lim_{J \rightarrow \infty} \frac{1}{J} \Sigma x_i \right) \left(\lim_{J \rightarrow \infty} \frac{1}{J} \Sigma (a_i - \bar{a})^2 \right) = - \bar{x} V (a)$$

But

$$V (a) = \frac{\sigma^2}{JR (x)}$$

thus,

$$\text{Cov} (a, b) = \sigma_{ab}^2 = - \frac{\sigma^2}{JR (x)}$$

APPENDIX G

DERIVATION OF DYNAMIC AND STATIC FATIGUE MEDIAN VALUE ERROR LIMITS NOT ASSUMING A PARTICULAR DISTRIBUTION

Dynamic Fatigue

$$\ln \hat{S} = \left(\frac{1}{N+1} \right) \ln \dot{\sigma} + \left(\frac{1}{N+1} \right) [\ln B + \ln (N+1) + (N-2) \ln \hat{S}_i]$$

$$y = a_1 x + b_1$$

$$a_1 = \frac{1}{N+1} , \quad b_1 = \left(\frac{1}{N+1} \right) [\ln B + \ln (N+1) + (N-2) \ln \hat{S}_i]$$

$$y = \ln \hat{S} , \quad x = \ln \dot{\sigma}$$

given $V(a_1)$, $V(b_1)$, and $Cov(a_1, b_1)$ from equations (20) through (22).

Solving for N in the equation for a_1 gives

$$N = \frac{1}{a_1} - 1$$

Applying the error-propagation rule,

$$V(N) = \left(\frac{\partial N}{\partial a_1} \right)^2 V(a_1) = \left(\frac{1}{a_1} \right)^4 V(a_1)$$

Solving for B in the equation for b_1 gives

$$\ln B = (N+1) b_1 - \ln (N+1) - (N-2) \ln \hat{S}_i = \frac{b_1}{a_1} + \ln a_1 - \left(\frac{1}{a_1} - 3 \right) \ln \hat{S}_i$$

Applying the error-propagation rule,

$$\begin{aligned} V(\ln B) &= \left(\frac{\partial \ln B}{\partial a_1} \right)^2 V(a_1) + \left(\frac{\partial \ln B}{\partial b_1} \right)^2 V(b_1) + 2 \left(\frac{\partial \ln B}{\partial a_1} \right) \left(\frac{\partial \ln B}{\partial b_1} \right) Cov(a_1, b_1) \\ &\quad + \left(\frac{\partial \ln B}{\partial \ln \hat{S}_i} \right)^2 V(\ln \hat{S}_i) \\ &= \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right)^2 V(a_1) + \left(\frac{1}{a_1} \right)^2 V(b_1) \\ &\quad + 2 \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right) \left(\frac{1}{a_1} \right) Cov(a_1, b_1) \\ &\quad + \left(\frac{1}{a_1} - 3 \right)^2 V(\ln \hat{S}_i) \end{aligned}$$

Strictly speaking, it is not possible to evaluate $V(\ln \hat{S}_i)$ without either assuming a particular distribution or going through a parameter analysis such as the one performed for the Weibull distribution in reference 3 or else testing many samples of specimens. However, since $V(\ln \hat{S}_i)$ is probably close to $V(\ln \bar{S}_i)$ for most distributions and since $V(\ln \bar{S}_i) = (1/J_1) V(\ln S_i)$ (Bevington, reference 20, page 21), the equivalence of $V(\ln \hat{S}_i)$ and $V(\ln \bar{S}_i)$ is assumed so that a particular distribution does not need to be specified when using only measured data to define $V(\ln B)$. Thus,

$$\begin{aligned} V(\ln B) = & \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right)^2 V(a_1) + \left(\frac{1}{a_1} \right)^2 V(b_1) \\ & + 2 \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right) \left(\frac{1}{a_1} \right) \text{Cov}(a_1, b_1) \\ & + \frac{1}{J_1} \left(\frac{1}{a_1} - 3 \right)^2 V(\ln S_i) \end{aligned}$$

Applying the differential expression for covariance,

$$\text{Cov}(N, \ln B) = \frac{d \ln B}{dN} V(N) = \left(\frac{\partial \ln B}{\partial N} \frac{\partial N}{\partial N} + \frac{\partial \ln B}{\partial b_1} \frac{\partial b_1}{\partial N} + \frac{\partial \ln B}{\partial \ln \hat{S}_i} \frac{\partial \ln \hat{S}_i}{\partial N} \right) V(N)$$

$$\frac{\partial \ln B}{\partial N} = \left(b_1 - \frac{1}{N+1} - \ln \hat{S}_i \right), \quad \frac{\partial N}{\partial N} = 1$$

$$\frac{\partial \ln B}{\partial b_1} = N+1, \quad \frac{\partial b_1}{\partial N} = \frac{db_1}{dN} - \frac{1}{N-1} \left(b_1 - \frac{1}{N+1} - \ln \hat{S}_i \right) \quad (\text{Note 1})$$

$$\frac{\partial \ln B}{\partial \ln \hat{S}_i} = -(N-2), \quad \frac{\partial \ln \hat{S}_i}{\partial N} = 0$$

$$\text{Cov}(N, \ln B) = (N+1) \frac{db_1}{dN} V(N) = (N+1) \text{Cov}(b_1, N)$$

however,

$$\text{Cov}(b_1, N) = -\frac{1}{a_1^2} \text{Cov}(a_1, b_1) \quad (\text{Note 2})$$

Thus,

$$\text{Cov}(N, \ln B) = -(N-1) \left(\frac{1}{a_1^2} \right) \text{Cov}(a_1, b_1)$$

Note 1:

$$\frac{\partial b_1}{\partial N} = \frac{db_1}{dN} - \left(\frac{1}{N+1} \right) \left(b_1 - \frac{1}{N+1} - \ln \hat{S}_i \right)$$

is obtained from the expression for the total derivative of $b_1 = b_1(N, \ln B, \ln \hat{S}_i)$

$$\begin{aligned}\frac{db_1}{dN} &= \frac{\partial b_1}{\partial N} \frac{\partial N}{\partial N} + \frac{\partial b_1}{\partial \ln B} \frac{\partial \ln B}{\partial N} + \frac{\partial \ln B}{\partial \ln \hat{S}_i} \frac{\partial \ln \hat{S}_i}{\partial N} \\ &= \frac{\partial b_1}{\partial N} + \left(\frac{1}{N+1} \right) \left(b_1 - \frac{1}{N+1} - \ln \hat{S}_i \right) \text{ as } \frac{\partial \ln \hat{S}_i}{\partial N} = 0\end{aligned}$$

This differs from Ritter et al. (reference 3). On page 804 of that reference, the authors indicate that $\partial b_1 / \partial N = db_1 / dN$, which is true for an intercept that depends only on N , but in this case there is a dependence on $\ln B$ and $\ln \hat{S}_i$ that, strictly speaking, should not be left out. The result is that the terms $\ln B (N+1)$ and $-3 \ln \hat{S}_i$ in equation (A43) of that reference drop out of the expression for covariance when a Weibull distribution is assumed as shown in Appendix H.

Note 2:

$Cov(b_1, N) = -1/a_1^2 Cov(a_1, b_1)$ because applying the error-propagation rule to the expression for $\ln B$ as a function of a_1, b_1 , and $\ln \hat{S}_i$; and applying it again for $\ln B$ as a function of N, b_1 , and $\ln \hat{S}_i$; and comparing terms show that

$$2 \left(\frac{\partial \ln B}{\partial a_1} \right) \left(\frac{\partial \ln B}{\partial b_1} \right) Cov(a_1, b_1) = 2 \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right) \left(\frac{1}{a_1} \right) Cov(a_1, b_1)$$

must equal

$$\begin{aligned}2 \left(\frac{\partial \ln B}{\partial b_1} \right) \left(\frac{\partial \ln B}{\partial N} \right) Cov &= (b_1, N) = 2 (N+1) \left(b_1 - \frac{1}{N+1} - \ln \hat{S}_i \right) Cov(b_1, N) \\ &= 2 \left(\frac{1}{a_1} \right) (b_1 - a_1 - \ln \hat{S}_i) Cov(b_1, N)\end{aligned}$$

which means

$$Cov(b_1, N) = -\frac{1}{a_1^2} Cov(a_1, b_1)$$

Static Fatigue

$$\ln \hat{t}_f = -N \ln \sigma_a + \ln B + (N-2) \ln \hat{S}_i$$

$$y = a_2 x + b_2$$

$$a_2 = -N, \quad b_2 = \ln B + (N-2) \ln \hat{S}_i$$

$$y = \ln \hat{t}_f, \quad x = \ln \sigma_a$$

given $V(a_2)$, $V(b_2)$, and $Cov(a_2, b_2)$ from equations (20) through (22).

Solving for N in the equation for a_2 gives

$$N = -a_2$$

Applying the error-propagation rule,

$$V(N) = \left(\frac{\partial N}{\partial a_2} \right)^2 V(a_2) = V(a_2)$$

Solving for $\ln B$ in the equation for b_2 gives

$$\ln B = b_2 - (N-2) \ln \hat{S}_i = b_2 + (a_2 + 2) \ln \hat{S}_i$$

Applying the error-propagation rule,

$$\begin{aligned} V(\ln B) &= \left(\frac{\partial \ln B}{\partial a_2} \right)^2 V(a_2) + \left(\frac{\partial \ln B}{\partial b_2} \right)^2 V(b_2) + 2 \left(\frac{\partial \ln B}{\partial a_2} \right) \left(\frac{\partial \ln B}{\partial b_2} \right) \text{Cov}(a_2, b_2) \\ &\quad + \left(\frac{\partial \ln B}{\partial \ln \hat{S}_i} \right)^2 V(\ln \hat{S}_i) \\ &= (\ln \hat{S}_i)^2 V(a_2) + V(b_2) + 2 (\ln \hat{S}_i) \text{Cov}(a_2, b_2) \\ &\quad + (a_2 + 2)^2 V(\ln \hat{S}_i) \end{aligned}$$

Again, assuming the equivalence of $V(\ln \hat{S}_i)$ and $V(\ln \bar{S}_i)$, as previously discussed for dynamic fatigue, results in

$$\begin{aligned} V(\ln B) &= (\ln \hat{S}_i)^2 V(a_2) + V(b_2) + 2 (\ln \hat{S}_i) \text{Cov}(a_2, b_2) \\ &\quad + (a_2 + 2)^2 \frac{1}{J_1} V(\ln S_i) \end{aligned}$$

Applying the differential expression for covariance,

$$\text{Cov}(N, \ln B) = \frac{d \ln B}{dN} V(N) = \left(\frac{\partial \ln B}{\partial N} \frac{\partial N}{\partial N} + \frac{\partial \ln B}{\partial b_2} \frac{\partial b_2}{\partial N} + \frac{\partial \ln B}{\partial \ln \hat{S}_i} \frac{\partial \ln \hat{S}_i}{\partial N} \right) V(N)$$

$$\frac{\partial \ln B}{\partial N} = -\ln \hat{S}_i, \quad \frac{\partial N}{\partial N} = 1$$

$$\frac{\partial \ln B}{\partial b_2} = 1, \quad \frac{\partial b_2}{\partial N} = \left(\frac{db_2}{dN} + \ln \hat{S}_i \right) \quad (\text{Note 3})$$

$$\frac{\partial \ln B}{\partial \ln \hat{S}_i} = -(N-2), \quad \frac{\partial \ln \hat{S}_i}{\partial N} = 0$$

$$\text{Cov}(N, \ln B) = \frac{db_2}{dN} V(N) = \text{Cov}(b_2, N)$$

but

$$\text{Cov}(b_2, N) = -\text{Cov}(a_2, b_2) \quad (\text{Note 4})$$

therefore,

$$\text{Cov}(N, \ln B) = -\text{Cov}(a_2, b_2)$$

Note 3:

$$\begin{aligned}\frac{db_2}{dN} &= \frac{\partial b_2}{\partial N} \frac{\partial N}{\partial N} + \frac{\partial b_2}{\partial \ln B} \frac{\partial \ln B}{\partial N} + \frac{\partial b_2}{\partial \ln \hat{S}_i} \frac{\partial \ln \hat{S}_i}{\partial N} \\ &= \frac{\partial b_2}{\partial N} - \ln \hat{S}_i \quad \text{as} \quad \frac{\partial \ln \hat{S}_i}{\partial N} = 0\end{aligned}$$

As with dynamic fatigue, using this expression for the total derivative to evaluate the partial derivative results in the term $-\ln \hat{S}_i$ dropping out of equation (A54) in reference 3 as shown in Appendix H.

Note 4:

From applying the error-propagation rule to $\ln B = \ln B(a_2, b_2, \ln \hat{S}_i)$ and to $\ln B = \ln B(N, b_2, \ln \hat{S}_i)$,

$$2 \left(\frac{\partial \ln B}{\partial a_2} \right) \left(\frac{\partial \ln B}{\partial b_2} \right) \text{Cov}(a_2, b_2) = \ln \hat{S}_i \text{Cov}(a_2, b_2)$$

must equal

$$2 \left(\frac{\partial \ln B}{\partial b_2} \right) \left(\frac{\partial \ln B}{\partial N} \right) \text{Cov}(b_2, N) = (-\ln \hat{S}_i) \text{Cov}(b_2, N)$$

APPENDIX H

DERIVATION OF DYNAMIC AND STATIC FATIGUE HOMOLOGOUS STRESS RATIO ERROR LIMITS NOT ASSUMING A PARTICULAR DISTRIBUTION

Dynamic Fatigue

$$\ln \sigma_{HD} = \left(\frac{1}{N+1} \right) \ln \frac{\dot{\sigma}}{S_i^3} + \frac{1}{N+1} [\ln B + \ln (N+1)]$$

$$y = a_3 x + b_3$$

$$a_3 = \frac{1}{N+1} \quad , \quad b_3 = \frac{1}{N+1} [\ln B + \ln (N+1)]$$

$$y = \ln \sigma_{HD} \quad , \quad x = \ln \frac{\dot{\sigma}}{S_i^3}$$

given $V(a_3)$, $V(b_3)$, and $Cov(a_3, b_3)$ from equations (20) through (22).

Solving for N in the equation for a_3 gives

$$N = \frac{1}{a_3} - 1$$

Applying the error-propagation rule,

$$V(N) = \left(\frac{\partial N}{\partial a_3} \right)^2 V(a_3) = \left(\frac{1}{a_3^2} \right) V(a_3)$$

Solving for $\ln B$ in the expression for b_3 gives

$$\ln B = (N+1) b_3 - \ln (N+1) = \frac{b_3}{a_3} + \ln a_3$$

Applying the error-propagation rule,

$$\begin{aligned} V(\ln B) &= \left(\frac{\partial \ln B}{\partial a_3} \right)^2 V(a_3) + \left(\frac{\partial \ln B}{\partial b_3} \right)^2 V(b_3) + 2 \left(\frac{\partial \ln B}{\partial a_3} \right) \left(\frac{\partial \ln B}{\partial b_3} \right) Cov(a_3, b_3) \\ &= \left(-\frac{b_3}{a_3^2} + \frac{1}{a_3} \right)^2 V(a_3) + \left(\frac{1}{a_3} \right)^2 V(b_3) + 2 \left(-\frac{b_3}{a_3^2} + \frac{1}{a_3} \right) \left(\frac{1}{a_3} \right) Cov(a_3, b_3) \end{aligned}$$

Applying the differential expression for covariance,

$$Cov(N, \ln B) = \frac{d \ln B}{dN} V(N) = \left(\frac{\partial \ln B}{\partial N} \frac{\partial N}{\partial a_3} + \frac{\partial \ln B}{\partial b_3} \frac{\partial b_3}{\partial a_3} \right) V(N)$$

$$\frac{\partial \ln B}{\partial N} = \left(b_3 - \frac{1}{N+1} \right) \quad , \quad \frac{\partial N}{\partial a_3} = -1$$

$$\frac{\partial \ln B}{\partial b_3} = (N+1) \quad , \quad \frac{\partial b_3}{\partial a_3} = \frac{\partial b_3}{\partial N} - \left(\frac{1}{N+1} \right) \left(b_3 - \frac{1}{N+1} \right) \quad (\text{Note 1})$$

$$\text{Cov} (N, \ln B) = \frac{db_3}{dN} V(N) = \text{Cov} (b_3, N)$$

but

$$\text{Cov} (b_3, N) = -\frac{1}{a_3^2} \text{Cov} (a_3, b_3) \quad (\text{Note 2})$$

Therefore,

$$\text{Cov} (N, \ln B) = -\frac{1}{a_3^2} \text{Cov} (a_3, b_3)$$

Note 1:

$$\frac{db_3}{dN} = \frac{\partial b_3}{\partial N} \frac{\partial N}{\partial N} + \frac{\partial b_3}{\partial \ln B} \frac{\partial \ln B}{\partial N} = \frac{\partial b_3}{\partial N} + \left(\frac{1}{N+1} \right) \left(b_3 - \frac{1}{N+1} \right)$$

Note 2:

$$2 \left(\frac{\partial \ln B}{\partial a_3} \right) \left(\frac{\partial \ln B}{\partial b_3} \right) \text{Cov} (a_3, b_3) = 2 \left(-\frac{b_3}{a_3^2} + \frac{1}{a_3} \right) \left(\frac{1}{a_3} \right) \text{Cov} (a_3, b_3)$$

equals

$$\begin{aligned} 2 \left(\frac{\partial \ln B}{\partial b_3} \right) \left(\frac{\partial \ln B}{\partial N} \right) \text{Cov} (b_3, N) &= 2 (N+1) \left(b_3 - \frac{1}{N+1} \right) \text{Cov} (b_3, N) \\ &= 2 \left(\frac{1}{a_3} \right) (b_3 - a_3) \text{Cov} (b_3, N) \end{aligned}$$

Static Fatigue

$$\ln (t_f S_i^2) = -N \ln \sigma_{HS} + \ln B$$

$$y = a_4 x + b_4$$

$$a_4 = -N, \quad b_4 = \ln B$$

$$y = \ln (t_f S_i^2), \quad x = \ln \sigma_{HS}$$

given $V(a_4)$, $V(b_4)$, and $\text{Cov}(a_4, b_4)$ from equations (20) through (22).

Solving for N in the equation for a_4 gives

$$N = -a_4$$

Applying the error-propagation rule,

$$V(N) = V(a_4)$$

Solving for $\ln B$ in the expression for b_4 gives

$$\ln B = b_4$$

Applying the error-propagation rule,

$$V(\ln B) = V(b_4)$$

Applying the differential expression for covariance,

$$\text{Cov} (N, \ln B) = \frac{d \ln B}{dN} V (N) = \left(\frac{\partial \ln B}{\partial b_4} \frac{\partial b_4}{\partial N} \right) V (N) = \frac{\partial b_4}{\partial N} V (N)$$

But in this case

$$\frac{\partial b_4}{\partial N} = \frac{db_4}{dN}$$

thus,

$$\text{Cov} (N, \ln B) = \frac{db_4}{dN} V (N) = \text{Cov} (b_4, N) \quad ,$$

but

$$\text{Cov} (b_4, N) = - \text{Cov} (a_4, b_4)$$

thus,

$$\text{Cov} (N, \ln B) = - \text{Cov} (a_4, b_4)$$

APPENDIX I

DYNAMIC AND STATIC FATIGUE ERROR LIMITS ASSUMING WEIBULL DISTRIBUTIONS

(after reference 3)

Dynamic Fatigue

From Appendix F,

$$V(N) = \frac{1}{a_1^4} V(a_1) = (N+1)^4 V(a_1)$$

but from equation (A38), page 805, reference 3, assuming Weibull distributions for S and S_i and using x_F as an independent variable results in

$$V(a_1) = \frac{K_d(N-2)^2}{J_0(N+1)^2 m_i R(\ln \dot{\sigma})}$$

where

$K_d = 1.44$ for most glasses and ceramics (Note 1)

J_1 = the number of specimens tested at each of J_2 stressing rates $\dot{\sigma}_j$ for
a total number of specimens $J_0 = J_1 J_2$

$$R(\ln \dot{\sigma}) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \dot{\sigma}_j - \overline{\ln \dot{\sigma}})$$

$$\overline{\ln \dot{\sigma}} = \frac{1}{J_2} \sum_{j=1}^{J_2} \ln \dot{\sigma}_j$$

Therefore,

$$V(N) = \frac{K_d(N-2)^2(N+1)^2}{J_0 m_i^2 R(\ln \dot{\sigma})}$$

Note 1:

$$K_d = m_i^2 \left(\frac{N+1}{N-2} \right)^2 \frac{\Gamma \left[1 + \frac{2}{m_i} \left(\frac{N-2}{N+1} \right) \right] - \Gamma^2 \left[1 + \frac{N-2}{m_i(N+1)} \right]}{\Gamma^2 \left[1 + \frac{N-2}{m_i(N+1)} \right]} \\ + \left\{ \ln \ln 2 - m_i \left(\frac{N+1}{N-2} \right) \ln \Gamma \left[1 + \frac{N-2}{m_i(N+1)} \right] \right\}^2$$

Γ = the gamma function

Similarly, from Appendix F, using equations (21) and (22) to substitute for $V(b_1)$ and $Cov(a_1, b_1)$,

$$\begin{aligned}
V(\ln B) &= \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right)^2 V(a_1) + \left(\frac{1}{a_1} \right)^2 \left[R(\ln \hat{\sigma}) + \overline{(\ln \hat{\sigma})^2} \right] V(a_1) \\
&\quad - 2 \left(-\frac{b_1}{a_1^2} + \frac{1}{a_1} + \frac{1}{a_1^2} \ln \hat{S}_i \right) \left(\frac{1}{a_1} \right) \overline{\ln \hat{\sigma}} V(a_1) + \left(\frac{1}{a_1} - 3 \right)^2 V(\ln \hat{S}_i) \\
&= \frac{1}{a_1^4} (-b_1 + a_1 + \ln \hat{S}_i)^2 V(a_1) + \frac{1}{a_1^2} [R(\ln \hat{\sigma}) + \overline{(\ln \hat{\sigma})^2}] V(a_1) \\
&\quad - 2 \left(\frac{1}{a_1^3} \right) (-b_1 + a_1 + \ln \hat{S}_i) \overline{\ln \hat{\sigma}} V(a_1) + \left(\frac{1}{a_1} - 3 \right)^2 V(\ln \hat{S}_i) \\
&= \frac{V(a_1)}{a_1^2} \left\{ \frac{1}{a_1^2} (-b_1 + a_1 + \ln \hat{S}_i)^2 + [R(\ln \hat{\sigma}) + \overline{(\ln \hat{\sigma})^2}] \right. \\
&\quad \left. - \frac{2}{a_1} (-b_1 + a_1 + \ln \hat{S}_i) \overline{\ln \hat{\sigma}} \right\} + \left(\frac{1}{a_1} - 3 \right)^2 V(\ln \hat{S}_i)
\end{aligned}$$

But from equations (A29) and (A35) in reference 3,

$$V(\ln \hat{S}_i) = \frac{K_i}{J_1 m_i^2} \approx \frac{K_d}{J_1 m_i^2}$$

Expanding the last term in the expression for $V(\ln B)$,

$$\begin{aligned}
\left(\frac{1}{a_1} - 3 \right)^2 V(\ln \hat{S}_i) &= \left(\frac{N+1}{N-2} \right)^2 \left(\frac{N-2}{N+1} \right)^2 \frac{1}{a_1^2} (1 - 3a_1^2) \frac{K_d}{J_1 m_i^2} \frac{R(\ln \hat{\sigma})}{R(\ln \hat{\sigma})} \frac{J_2}{J_2} \\
&= J_2 \left(\frac{N+1}{N-2} \right)^2 (1 - 3a_1^2) R(\ln \hat{\sigma}) \frac{1}{a_1^2} V(a_1)
\end{aligned}$$

Thus,

$$\begin{aligned}
V(\ln B) &= \frac{V(a_1)}{a_1^2} \left\{ \frac{1}{a_1^2} (-b_1 + a_1 + \ln \hat{S}_i)^2 + [R(\ln \hat{\sigma}) + \overline{(\ln \hat{\sigma})^2}] - \frac{2}{a_1} (-b_1 + a_1 + \ln \hat{S}_i) \overline{\ln \hat{\sigma}} \right. \\
&\quad \left. + J_2 \left(\frac{N+1}{N-2} \right)^2 (1 - 3a_1^2) R(\ln \hat{\sigma}) \right\} \\
&= \frac{V(a_1)}{a_1^2} \left\{ \left(\frac{-b_1 + a_1 + \ln \hat{S}_i}{a_1} - \overline{\ln \hat{\sigma}} \right)^2 + R(\ln \hat{\sigma}) \left[1 + J_2 \left(\frac{N+1}{N-2} \right)^2 (1 - 3a_1^2) \right] \right\}
\end{aligned}$$

Substituting for b_1 and a_1 gives

$$V(\ln B) = \frac{V(a_1)}{a_1^2} \left\{ \left[-\ln B - \ln(N+1) + 3 \ln \hat{S}_i + 1 - \overline{\ln \hat{\sigma}} \right] + R(\ln \hat{\sigma}) \left[1 + J_2 \left(\frac{N+1}{N-2} \right)^2 \left(1 - \frac{3}{N+1} \right)^2 \right] \right\}$$

The factors $\left(\frac{N+1}{N-2}\right)^2 \left(1 - \frac{3}{N+1}\right)^3$ are approximately 1, which reduces the expression to equation (A42) in reference 3,

$$V(\ln B) = \frac{K_d (N-2)^2}{J_0 m_i^2 R (\ln \dot{\sigma})} \left\{ \left[3 \ln \hat{S}_i + 1 - \ln [B (N+1)] - \overline{\ln \dot{\sigma}} \right]^2 + (1 + J_2) R (\ln \dot{\sigma}) \right\}$$

To evaluate the covariance term, the expression for covariance derived in Appendix F is used, which gives

$$\text{Cov}(N, \ln B) = -(N+1) \left(\frac{1}{a_1^2} \right) \text{Cov}(a_1, b_1) = -(N+1)^3 \text{Cov}(a_1, b_1)$$

but

$$\text{Cov}(a_1, b_1) = -V(a_1) \overline{\ln \dot{\sigma}} = -\frac{K_d (N+2)^2 \overline{\ln \dot{\sigma}}}{J_0 (N+1)^2 m_i^2 R (\ln \dot{\sigma})}$$

Therefore,

$$\text{Cov}(N, \ln B) = \frac{K_d (N-2)^2 (N+1) \overline{\ln \dot{\sigma}}}{J_0 m_i^2 R (\ln \dot{\sigma})}$$

Static Fatigue

From Appendix F,

$$V(N) = V(a_2)$$

but from equation (A49), reference 3,

$$V(a_2) = \frac{K_s (N-2)^2}{J_0 m_i^2 R (\ln \sigma_a)} = V(N)$$

where

$$K_s = \frac{m_i^2}{(N-2)^2} \frac{\Gamma \left[1 + \frac{2(N-2)}{m_i} \right] - \Gamma^2 \left(1 + \frac{N-2}{m_i} \right)}{\Gamma^2 \left(1 + \frac{N-2}{m_i} \right)} + \left[\ln \ln 2 - \frac{m_i}{N-2} \ln \Gamma \left(1 + \frac{N-2}{m_i} \right) \right]^2$$

J_1 = the number of specimens tested at each of J_2 applied stresses, σ_{aj} , for a total number of specimens $J_0 = J_1 J_2$

$$R(\ln \sigma_a) = \frac{1}{J_2} \sum_{j=1}^{J_2} (\ln \sigma_{aj} - \overline{\ln \sigma_a})$$

$$\overline{\ln \sigma_a} = \frac{1}{J_2} \sum_{j=1}^{J_2} \ln \sigma_{aj}$$

Similarly, from Appendix F,

$$\begin{aligned}
V(\ln B) &= (\ln \hat{S}_i)^2 V(a_2) + V(b_2) + 2(V \ln \hat{S}_i) \text{Cov}(a_2, b_2) + (a_2 + 2)^2 V(\ln \hat{S}_i) \\
&= (\ln \hat{S}_i)^2 V(a_2) + [R(\ln \sigma_a) + \overline{\ln \sigma_a^2}] - 2 \ln \hat{S}_i \overline{\ln \sigma_a} V(a_2) \\
&\quad + (a_2 + 2)^2 V(\ln \hat{S}_i)
\end{aligned}$$

but

$$V(\ln \hat{S}_i) = \frac{K_i}{J_1 m_i^2}$$

where

$$K_i \approx 1.44 \quad (\text{Note 2})$$

Thus,

$$V(\ln B) = \frac{K_s (N-2)^2}{J_0 m_i^2 R(\ln \sigma_a)} [(\ln S_i - \overline{\ln \sigma_a})^2 + R(\ln \sigma_a)] + \frac{(N-2)^2 K_i}{J_1 m_i^2}$$

To evaluate the covariance term, the expression from Appendix F is used, which gives

$$\begin{aligned}
\text{Cov}(N, \ln B) &= -\text{Cov}(a_2, b_2) = -(-V(a_2) \overline{\ln \sigma_a}) = V(a_2) \overline{\ln \sigma_a} \\
\text{Cov}(N, \ln B) &= -\frac{K_s (N-2)^2 \overline{\ln \sigma_a}}{J_0 m_i^2 R(\ln \sigma_a)}
\end{aligned}$$

Note 2:

$$K_i = m_i^2 \frac{\Gamma(1 + 2/m_i) - \Gamma^2(1 + 1/m_i)}{\Gamma^2(1 + 2/m_i)} + [\ln \ln 2 - m_i \ln \Gamma(1 + 1/m_i)]^2$$

1. Report No. NASA RP-1174	2. Government Accession No.	3. Recipient's Catalog No.	
4. Title and Subtitle A Method for Developing Design Diagrams for Ceramic and Glass Materials Using Fatigue Data		5. Report Date September 1986	
		6. Performing Organization Code 313	
7. Author(s) T. M. Heslin, M. B. Magida,* and K. A. Forrest		8. Performing Organization Report No. G-86F0007	
9. Performing Organization Name and Address Goddard Space Flight Center Greenbelt, Maryland 20771		10. Work Unit No.	
		11. Contract or Grant No.	
		13. Type of Report and Period Covered Reference Publication	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546		14. Sponsoring Agency Code	
15. Supplementary Notes *Presently with Optical Group Research, Perkin Elmer, Danbury, Connecticut 06810			
16. Abstract The service lifetime of glass and ceramic materials can be expressed as a plot of time-to-failure versus applied stress whose plot is parametric in percent probability of failure. This type of plot is called a design diagram. Confidence interval estimates for such plots depend on the type of test that is used to generate the data, on assumptions made concerning the statistical distribution of the test results, and on the type of analysis used. This report outlines the development of design diagrams for glass and ceramic materials in engineering terms using static or dynamic fatigue tests, assuming either no particular statistical distribution of test results or a Weibull distribution and using either median value or homologous ratio analysis of the test results.			
17. Key Words (Selected by Author(s)) Quality assurance, Materials, Glasses, Ceramics, Fatigue, Delayed brittle failure		18. Distribution Statement Unclassified - Unlimited STAR Category 38	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages 58	22. Price* A04

*For sale by the National Technical Information Service, Springfield, Virginia 22151.

National Aeronautics and
Space Administration
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